Title	State Estimation and Control for Nonlinear Stochastic Systems(Dissertation_全文)
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Citation	Kyoto University (京都大学)
Issue Date	1976-05-24
URL	http://dx.doi.org/10.14989/doctor.r3073
Right	
Type	Thesis or Dissertation
Textversion	author

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CONTENTS

ACKNOWLEDGMENTS

It is with a great deal of pleasure that I acknowledge the critical review of the manuscript and helpful suggestions by Professor Yoshifumi Sunahara of Kyoto Institute of Technology who played an important role to stimulate the research works developed in this dissertation. I also express my sincere appreciation for his continuous encouragement and valuable support. Thanks should be extended to K.Yamashita, Y.Matsukura, S.Sugimoto, H.Torii, N.Ohse, H.Kano, Y.Sakata, M.Yamashita, K.Kameshima, M.Imamura, K.Ohnishi, Y.Mogami and Y.Takeuchi, and to many other graduate students who have generously provided me with stimulative discussions and helpful assistances.

This research was performed while I have been a Research Assistant at the Division of Control Sciences, Faculty of Polytechnic Sciences, Kyoto Institute of Technology. During the period, part of this research was supported by the Japan Educational Research Subsidy for Promotion of Science under Grant No.775116.

ABSTRACT

The problem of state estimation and control for a wide class of nonlinear stochastic lumped or distributed parameter systems under noisy observations is studied in the framework of Itô stochastic calculus. The purpose of this dissertation is to describe two important phases: to give mathematical developments for the theories of signal detection, filtering, parameter identification and control, and to show the algorithm of computer implementations for the scheme of control systems.

This dissertation is divided into two major parts. Part One is devoted to the approximate methods of state estimation and control for nonlinear systems described by the Itô stochastic differential equation, and Part Two is devoted to provide methods of state and parameter estimation and control for stochastic systems modeled by partial differential equations.

The basic notion of the proposed methods developed in Part One

is a use of the stochastic linearization technique to the field of nonlinear control systems. With this technique, a joint scheme of estimation and control is presented, emphasizing that the stochastic linearization method plays a useful role in realizing a stochastic optimal control system. Part One is divided mainly into four chapters: the first is concerned with the mathematical aspect of the models, terminology and a review of stochastic linearization technique which is necessary to understand the treatment of problems, the second a possible solution to the signal detection in Gaussian noise, the third some approximate versions of nonlinear filters in various situations, and the final a practical scheme for estimation-control, including the important aspect of sufficient statistics for the purpose of observation data reduction.

In Part Two, based on an extended version of Itô stochastic equation to the distributed parameter systems, the model of a control system is described by a stochastic nonlinear partial differential equation. By using such approximation techniques as Taylor series expansion and stochastic linearization extended to the distributed parameter system, estimation and control problems are solved. Part Two is divided into three main chapters: the first is concerned with the filtering problem, the second the parameter identification, and the third the problem of optimal control for a general class of linear distributed systems and extensively for a class of nonlinear distributed parameter systems.

Throughout the two parts of the dissertation, various kinds of numerical computations are performed in order to show the practical computer implementation.

I. PART ONE. APPROXIMATE METHODS OF STATE ESTIMATION AND CONTROL FOR NONLINEAR LUMPED PARAMETER SYSTEMS

CHAPTER 1. INTRODUCTION

Physical systems are, in general, designed and built to perform the minimization or the maximization of a preassigned cost functional. For example, aircrafts, spacecrafts, submarines and some vehicles must navigate in their respective environments to accomplish their missions. In order to know whether or not a system is performing suitably, and ultimately to control the system performance, the system designer must recognize the "state" of the system at any instant of time, where in navigation systems the state consists of position, velocity, acceleration, etc., of the craft in question. Physical systems are often subjected to random disturbances, so that the system state may itself be stochastic.* When the designer wishes to know the state at hand, he will take measurements or observations on the system through a measuring device. These measurements are generally contaminated with noise which is called as observation noise.

It is also an inevitable feature that a dynamical system to be controlled often exhibits various kinds of nonlinear characteristics.

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* The word "stochastic" comes from Greek "στοχαστικός" (to aim or to guess) and is used synonymously with the word "random."

Thus, for the system designer, the general problem to be solved is to find the control of a noisy nonlinear dynamical system in some optimal fashion, given only an incomplete knowledge of the system. Under such coupled constraints as the linearity of dynamical systems, noisy observations and desired criterion given by quadratic cost functionals, it has already been shown that the optimal control and estimation problems of the system state may be independently solved by the versions of the "separation theorem." However, this is not the case in general for the optimal control of nonlinear dynamical systems, but the combined problems of optimal control and estimation must be treated simultaneously.

Since the establishment of the precise scheme for the state estimation and the optimal control of nonlinear dynamical systems is almost impossible, in Part One, the author will establish an approximate method which will be shown to play an important role to realize a broad class of stochastic optimal control.

The part one will be divided into three major parts: first a part on the mathematical aspects is developed of the system models and terminology and some concepts necessary to understand the treatment of problems secondly, some approximate versions of a nonlinear filter in various situations, and the nonlinear filtering problem as well as relations of filtering to control theory; and finally, a practical schemes of estimation-control, including the aspect of signal detection problem and also the data reduction problem.

The part one is devoted to describe two important phases: first, to give detailed stochastic methods suitable for research workers who are interested in controlling a nonlinear system under noisy observations, and secondly, to show the algorithm of the whole scheme of the optimal control systems.

1.1. Historical Background

The historical background of this research is divided into three parts.

1.1.A. Filtering Problem

The problem of estimating a random signal process based upon information contained in an observation process is itself one of the basic contexts of classical and, still, of modern system theory. In the early 1940's, Kolmogorov[73] and Wiener[155] developed a systematic approach for providing an estimate of a random signal process on the basis of observation of the signal process additively corrupted by noise. Their key notion was dependent on the assumptions of stationarity, ergodicity, and knowledge of the entire past of observed process. Kolmogorov solved the discrete-time problem by "pre-whitening" of the data, while Wiener solved the continuous-time problem in the frequency domain employing "spectral factorization." The result of their investigations was the specification of the weighting function of the optimal estimator as a solution of the Wiener-Hopf equation, and these early works in filtering theory were responsible for many advances in the statistical design of control systems.

The next substantial development in the (linear) filtering was the work of Kalman (1960) [64], and Kalman and Bucy (1961) [69], under weaker assumptions than those made in the original Wiener problem —— that is, nonstationary, observations known within only a finite time interval in the past, and vector observations of vector processes. The theory is known as the Kalman-Bucy filtering, and has provided numerous applications in the mid-1960's. Such major applications of the theory are in the field of satellite orbit determination, submarine and aircraft navigation, and space flight, including the Ranger, Mariner, Pioneer and historical Apollo missions in the U.S.A.[18] However, the Kalman-Bucy filter is rigorously valid only for linear filtering, even though, heuristically, nonlinear extensions were developed successfully for orbit determination, fire control and space navigation programs.

Since the work of Kalman and Bucy, there have been many variations on the Kalman-Bucy theme; these variations and the relation of the Kalman-Bucy theory to the Wiener-Kolmogorov theory are summarized in the tutorial article of Kailath[57] and in the textbook of Sunahara[127].

Although the Kalman's filtering theory found immediate applications to the problems of orbital determination, navigation, etc., it was soon apparent from these applications that the linear assumption was not adequate for many situations. The original investigations in nonlinear filtering were undertaken independently by Stratonovich[121] in the Soviet Union and by Kushner[74,75] in the U.S.A., using descrete-time approximations, Bayes rule and limiting the arguments to obtain the stochastic equation for evolution of the conditional density of the message (signal) process relative to the observation process. Much of the subsequent theoretical work in nonlinear continuous filtering was done by Kushner[78, 79] using Itô stochastic calculus. Bucy[16] introduced a representation theorem from which Kushner's result[74] can be derived and has provided significant generalizations of the theory of nonlinear filtering. This approach to continuous filtering was also taken by Wonham[158]. The results of Stratonovich[121] and Wonham[158] should be interpreted in the sense of Stratonovich for the stochastic calculus.

In the Soviet Union, since the early work of Stratonovich, several investigations have also worked on the theory of nonlinear filtering, notably Liptser and Shiryaev[88,89,115,116]. These works have been concerned with finding the stochastic equations for the conditional density function, similar to those by Wonham[158] and Kushner[78].

The probabilistic approach to nonlinear filtering which was used by Stratonovich, Kushner and by Wonham is based on the so-called Bayesian approach. Zakai (1969) [185] has introduced a method of nonlinear filtering with use of the transformation of a certain class of stochastic processes by absolutely continuous substitution of measures due to Girsanov[45] and has given a rigorous proof of the Bucy's representation theorem. In the Soviet Union, Ershov[34] also treats the related theoretical work.

1.1.B. Approximate Filter

Recognizing the importance of nonlinear filtering problems, various studies have been made by many investigators as surveyed in the previous subsection. The result reveals that an exact realization of optimal nonlinear filters requires infinite-dimensional filters which are practically almost impossible. In nonlinear filtering problems as well as in the linear ones, we are interested in computing the conditional mean and covariance matrix (these are the first- and second-moments respectively). Physically, the conditional mean is the minimum variance estimate, and the covariance matrix measures the uncertainty in the estimate.

Up to the present time, approximate schemes have been suggested on the physical realization of optimal nonlinear filters in an approximate form of finite dimensional filters; these trials are summarized in the textbook of Jazwinski [54, Chap.9]. The ideas of Kalman filter were extended to the estimation of the states of nonlinear dynamical systems using the so-called first-order, or extended Kalman filter (see, Ho and Lee[47], Cox[23], Mowery[99], Friedland and Bernstein[42], and others). In all of these papers different techniques such as least-squares, maximum-likelihood, etc., have been used to drive filter equations. Most of these techniques use a Taylor series expansion up to second-order terms, and derive linearized equations to compute the covariance matrix and the filter time-varying gains.

Using the stochastic calculus, the exact filter equations have been approximated to suboptimal finite-dimensional filters. Typical papers along this line of approach are those of Kushner[80], Bass et al.[6], Sorenson and Stubberud[120], etc. An suggestive approach was presented by Kushner[80] for approximation to the exact filter via moment sequences. The truncated second-order filter* was developed by Jazwinski [53], and independently by Bass et al.[6] Schwartz[111] and Fisher[36] independently developed the Gaussian second-order filter. In many of these works, second-order terms are retained in approximating the nonlinear functions. Sunahara[126] proposed to replace the nonlinear functions by quasi-linear functions via stochastic linearization. In this dissertation, such technique proposed by Sunahara will be extensively used to establish an overall system of estimation and control.

1.1.C. Control Problem

Starting about 1958, a new trend became established, stimulated partly by the rapidly increasing accessibility of digital computers and partly by the developing interest in particularly aerospace optimization problems. A branch of control theory has evolved largely within the framework of Bellman's "Dynamic Programming"[8] and "Adaptive

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* The approximate filter, which is derived under the assumption that third- and higher-order central moments are negligible, is called the "truncated second-order filter."

Control Processes"[9] which present a computer-oriented formulation of a large class of Markovian decision problems. By a series of celebrated papers by Kalman[64-69,71]], fundamental and essential researches were done on the concepts of state estimation, optimal control, system stability, controllability and observability in the control system theory. After these works by Kalman, using the stochastic calculus, the stochastic control theory has been developed mainly in the U.S.A. by Kushner[77,83], Wonham[161], Flemming[38,39] and many other researchers.

Because of the widespread use of linear filter and the demands for a construction of control systems, numerous papers have been written in a framework of the so-called linear-quadratic-Gaussian (LQG) context, celebrated by the "Separation Theorem" of Wonham[160]. Therefore it seems that the linear control theory has almost been established[70,77,83,161]. The excellent survey of the LQG problem is Ref.[97] in the special issue of IEEE Transaction on Automatic Control on the "Linear-Quadratic-Gaussian" Estimation and Control Problem" (vol.AC-16, no.6, Dec. 1971).

Although the LQG problem have reached a certain degree of maturity with respect to theoretical and algorithmic advances, on the other hand, there have been very few investigations to date of the problem of optimizing nonlinear stochastic systems. The control problem of nonlinear system is a current topics. Toward this, some of papers have appeared.

Kushner[76] presented a method of computing correction to the optimal deterministic control for the nonlinear systems where the effects of disturbance are small. Later, Kushner and Kleinman[84] considered several aspects of the numerical solution of the Bellman's optimization equation of nonlinear degenerate elliptic-type. A systematic procedure was given by Wonham and Cashman (1969) [162] for digital computation of a suboptimal nonlinear feedback control which is obtained by a combination of dynamic programming and statistical linearization for a class of time-invariant linear systems with amplitude bounded control. Alternatively, Smith and Man (1969) [119] developed a successive approximation technique based on statistical linearization for nonlinear time-invariant process under complete observations, and applied the technique to a chemical process example.

Independently, in 1969-1970, Sunahara and the author[129-131] developed an approximate method of estimation-control for a wide class of nonlinear stochastic systems via the stochastic linearization technique in Markovian framework. Shapiro and Mon[114] obtained the necessary conditions for the optimality of feedback gains for the one-dimensional nonlinear process whose dynamics and control are finite-degree polynomials with respect to the random variables via the method of expansion of the density function in an infinite series. Raja Rao and Mahalanabis discussed in [103] the results of application of the purturbation technique along with stochastic approximations, where the purtubation technique is combined with the statistical linearization in order to derive suboptimal solution. Also, in [104], by approximating nonlinear functions by second-order polynomials, Raja Rao and Mahalanabis obtained the suboptimal control for discrete-time systems with a special performance criterion function. A combined method of estimation and control was proposed by Dressler and Tabak[29], using the extended Kalman filter, and applied to satellite tracking system with the steady-state approximation.

Based on the Gaussian sum approximation to the *a posteriori* density function, Alspach[1] calculated certain suboptimal controls for discrete-time nonlinear systems. Recently, Tse et al.[146] considered the use of second-order terms and pertubation controls. The resulting control procedure is, however, too complicated to apply this technique to practical problems.

The above researches may be classified into the following major five categories:

(i) Statistical linearization method [104,119,162]

(ii) Stochastic linearization technique [29,129-131]

(iii) Approximation of probability density function [1,114]

(iv) Perturbation method [76,103,146]

(v) Numerical approach [84].

1.2. Problem Considered

We consider the problem of finding an optimal control for a class of nonlinear stochastic dynamical systems under noisy observations, and establish an approximate method of optimal control in a form of computer-oriented feedback control systems as might be expected. Our situation to

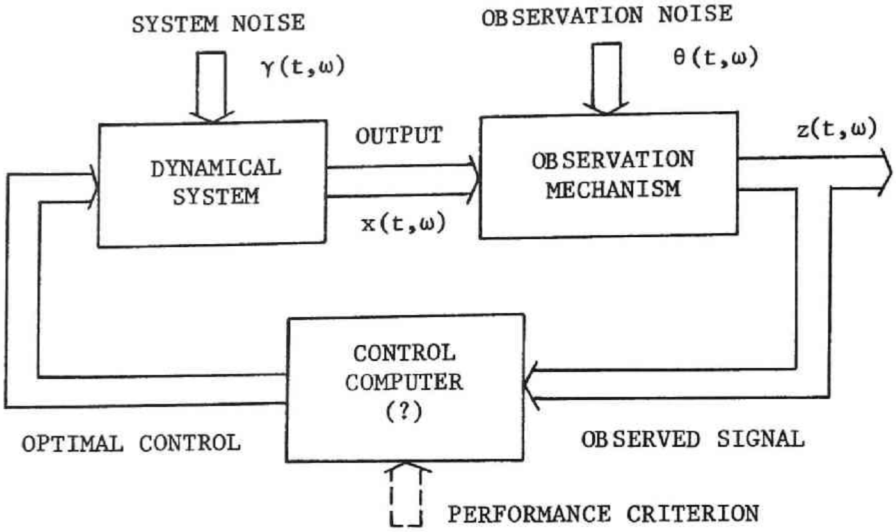

Fig.1.1. Problem illustration of optimal control under noisy observations.

control problem is shown in Fig.1.1. The dynamical system to be controlled under a given performance criterion is described by a vector nonlinear differential equation of dimension n.

$$\frac{dx(t,\omega)}{dt} = f[t,x(t,\omega)] + c[t,u(t)] + G[t,x(t,\omega)]\gamma(t,\omega), \quad t\varepsilon[t_0,T]. \tag{1.1}$$

In (1.1), $x(t,\omega)$ is an n-vector state variable; $f[t,x(t,\omega)]$ and $G[t,x(t,\omega)]$ are respectively an n-vector and an $n{\times}d_1$-matrix nonlinear function; $\gamma(t,\omega)$ is a d_1-vector white Gaussian noise with constant spectral density function*; $c[t,u(t)]$ is an n-vector forcing term; $u(t)$ is an m-dimensional control vector ($n \geq m$); and ω is the generic point of the probability space Ω.

The states of the system may not be able to be "completely" observed

* In most cases of the practical problems the system noise may not be "white" but "colored." However, for convenience of discussions and without loss of generality, we consider the white noise because the colored noise is easily whitened by introducing a suitable shaping filter.

because the output observation is sometimes corrupted by noise which is referred to the observation noise. The observation mechanism is given by

$$(1.2) \qquad z(t,\omega) = h[t,x(t,\omega)] + R(t)\theta(t,\omega).$$

The output $z(t,\omega)$ is an l-vector, where $l \leq n$; $h[t,x(t,\omega)]$ is an l-vector nonlinear function; $R(t)$ is an $l \times d_2$ parameter matrix; and $\theta(t,\omega)$ is a d_2-vector white Gaussian noise with unit power spectral density.

As will be pointed out in Chap.2, Sec.2.1, the mathematical models of both the dynamical system (1.1) and the observation mechanism (1.2) are purely formal because of the existence of white Gaussian noise terms. In order to make these models precise, we rewrite them as a couple of Itô stochastic differential equations,

$$(1.3) \qquad dx(t,\omega) = f[t,x(t,\omega)]dt + c[t,u(t,\omega)]dt + G[t,x(t,\omega)]dw(t,\omega)$$

$$(1.4) \qquad dy(t,\omega) = h[t,x(t,\omega)]dt + R(t)dv(t,\omega),$$

where newly introduced processes $w(t,\omega)$ and $v(t,\omega)$ are mutually independent Brownian motion processes, and $y(t,\omega)$ is an l-vector observation process which is related to $z(t,\omega)$ by the intuitive relation,

$$(1.5) \qquad z(t,\omega) = \dot{y}(t,\omega),$$

where the dot " · " denotes the differentiation with respect to time t.

In practical terms, our problem is to find a control vector $u(t)$ in such a way as to minimize the cost functional (performance criterion),

$$(1.6) \qquad J(u) = \mathcal{E}\{F[x(T),x^d(T)] + \int_{t_0}^{T} L[t,x(t),u(t)]dt\},$$

based on the *a priori* probability distribution of the initial state $x(t_0)$ where F and L are nonnegative scalar functions of the class $C^{(2)}$ and $x^d(T)$ is the desired state at final time T.

As already known, in order to solve the optimal control problem under noisy observations we must first solve the optimal filtering problem and then present the solution for the optimal control problem. Such a situation may be schematically shown as in Fig.1.2.

The important items to be emphasized in Part One are as follows:

(i) When we take the observation data, the data are always corrupted

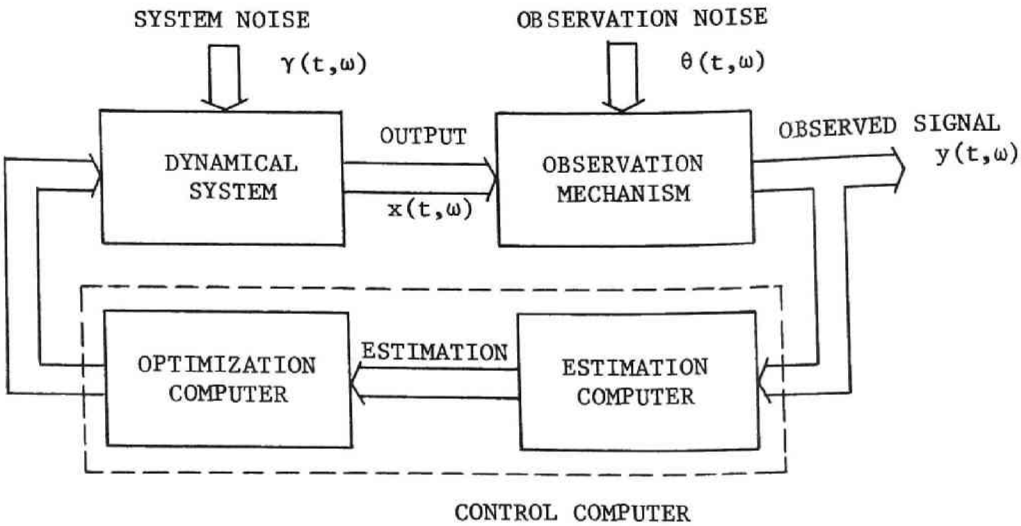

Fig.1.2. Overall configuration of optimal control for nonlinear dynamical systems under noisy observations.

additively by a random noise,

(ii) There exist various kinds of nonlinear characteristics in both the dynamical system and the observation mechanism.

Taking the item (i) into account, it is required to establish a procedure to solve the nonlinear filtering problem. Furthermore, from the item (ii) the possibility is no longer expected that the separation theorem[160] holds between state estimation and optimal control.

1.3. Summary of Contents

In constructing the physical control system, the avenue taken in this dissertation is first to establish a possible method of detection of signals in noise, and then to provide the approximate method of estimation based on the stochastic linearization technique, and finally to construct an overall scheme of joint estimation and control under a certain cost functional.

The outline of the part one is as follows.

In Chapter 2, some of general groundworks required in this study are presented as mathematical preliminaries. The precise mathematical models for the system are also established by the stochastic differential equations in the senses of Itô and Stratonovich.

As the stochastic linearization technique proposed by Sunahara[126] in Markovian framework is extensively used in the study, in Chapter 3 a brief review of the technique is given for better understanding, emphasizing an error evaluation and the discussions of relations between such a technique and the classical statistical equivalent linearization.

In Chapter 4, a new type of signal detection problem is formulated and its positive solution is proposed via a modified likelihood-ratio function. The signal detection problem in this chapter is to detect the true initial time from which the signal is surely present in the observation data to know what signal is transmitted. This situation leads us to the simultaneous signal detection and estimation problem.

Chapter 5 contains the development of the approximate filter equations, based on the stochastic linearization, for a wide class of nonlinear systems with state-independent and/or state-dependent noise or under state-dependent observation noise, respectively. A variety of digital simulation studies are also given with an analytical study for performance evaluation of the approximate filter dynamics.

Using the filter dynamics derived in Chapter 5, in Chapter 6 a successful and effective scheme to optimal control is presented, discussing some aspects of numerical approach.

In Chapter 7, in terms of the information state the important concept of sufficient statistics is discussed for the purpose of observation data reduction in stochastic control systems.

The remainder of Part One is devoted to discuss a summary of the results and some suggestions for areas of future researches.

CHAPTER 2. MATHEMATICAL PRELIMINARIES

2.0. Basic Definitions and Symbolic Conventions

Before presenting the key aspect of this dissertation, several basic definitions and symbolic conventions are presented.

Let $E^{(n)}$ denote an n-dimensional Euclidean space. If x is an element of $E^{(n)}$ ($x \varepsilon E^{(n)}$), then x' denotes the transpose of the vector x. Similarly, if M is a matrix, then M' denotes its transpose and $|M|$ denotes its determinant. As a rule, vector and matrix notations follow the usual manner, that is, lower case letters a, b and c,··· denote column vectors with i-th real components a_i, b_i and c_i,···. Capital letters A, B, C and D,··· denote matrices with elements a_{ij}, b_{ij}, c_{ij} and d_{ij},··· respectively. Certain algebraic quantities such as algebras, fields, etc. are expressed by the symbols, $\mathcal{S}$, $\mathcal{Y}$,···, etc.

The following background knowledges are important.[28,31,90,156]

(1) *Probability space:* Let Ω be a space of points ω, where Ω and ω are called the sample space and the generic point, respectively. Let

$\mathcal{S}$ be a σ-algebra of subsets of Ω. And let P be a probability measure on Ω, that is a measure which is normed, positive and σ-additive. The triplet $(\Omega,\mathcal{S},P)$ is called a probability space. The pair $(\Omega,\mathcal{S})$ is often referred to as a measurable space, and the pair $(\mathcal{S},P)$ is called a probability field.

(2) *Measurable function:* Let $(\Omega_1,\mathcal{S}_1)$ and $(\Omega_2,\mathcal{S}_2)$ be two measurable spaces, and let f be a function with domain Ω_1 and range in Ω_2. The function f is said to be a measurable function or a measurable mapping of $(\Omega_1,\mathcal{S}_1)$ into $(\Omega_2,\mathcal{S}_2)$ if for every set A in $\mathcal{S}_2$, the set

$$f^{-1}(A) = \{\omega: f(\omega)\varepsilon A\}$$

is in $\mathcal{S}_1$. The set $f^{-1}(A)$ is called the inverse image of A.

(3) *Random variable:* A real-valued function $x(\omega)$ defined on Ω is called a random variable if for every Borel set $\mathcal{B}$ in the Euclidean space $E^{(n)}$ the set $\{\omega: x(\omega)\varepsilon\mathcal{B}\}$ is in $\mathcal{S}$.

(4) *Expectation:* The expectation of the random variable x defined on a probability space $(\Omega,\mathcal{S},P)$ is given by

$$E\{x\} = \int_\Omega x dP.$$

(5) *Conditional expectation:* Let $(\Omega,\mathcal{S},P)$ be the basic probability space. Let $\mathcal{C}$ be a sub σ-algebra of $\mathcal{S}$. Let x be an integrable random function on Ω. The conditional expectation of x with respect to $\mathcal{C}$, denoted by $E\{x|\mathcal{C}\}$, is defined as any $\mathcal{C}$-measurable random variable satisfing

$$\int_C x dP = \int_C E\{x|\mathcal{C}\} dP$$

for all $C\varepsilon\mathcal{C}$.

(6) *Stochastic process:* A stochastic process $\{x(t,\omega),\ t\varepsilon T_0\}$ is a family of random variables, with a real parameter t and defined on the probability space $(\Omega,\mathcal{S},P)$.

For each t, $x(t,\omega)$ is an $\mathcal{S}$-measurable function. For each ω, $\{x(t,\omega), t\varepsilon T_0\}$ is a function defined on the parameter set T_0 and is called a sample function of the process. For economy of description, we omit to write the symbol ω in the following chapters in order to cause no confusion.

When a probability statement is true almost surely or true with probability 1, then the abbreviation a.s. or w.p.1 is used. A limit in the mean square is denoted by l.i.m.

A symmetric matrix A is positive definite if there exists a positive constant k such that for all $x \in E^{(n)}$

$$x'Ax > kx'x.$$

The Euclidean norm of an n-vector x is given by

$$\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2} = (x'x)^{1/2}$$

and for an n×m-matrix A

$$\|A\| = (\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2)^{1/2} = (\text{tr.}\{AA'\})^{1/2},$$

where "tr." denotes the trace of the matrix. If A is a symmetric, non-negative definite matrix, then we write $\|x\|_A^2 = x'Ax$ to denote the generalized Euclidean norm. The identity matrix is I. Notation $[\cdot]_{ij}$ expresses the (i,j)-component of a matrix. A real function f(x) is said to satisfy a Hölder condition with respect to λ, if for some constant k and all x and y,

$$\|f(x)-f(y)\| \leq k\|x-y\|^{\lambda}, \quad 0<\lambda\leq 1.$$

The symbol $\mathcal{Y}_t$ denotes the smallest σ-algebra of ω sets with respect to which the random variables $y(\tau)$ with $\tau \leq t$ are measurable. The conditional expectation of a random variable x(t) conditioned by $\mathcal{Y}_t$ is simply expressed by "$\hat{\ }$" such that $E\{x(t)|\mathcal{Y}_\tau\} = \hat{x}(t|\tau)$, where $\tau \leq t$.

For convenience of the present description, the principal symbols used here are listed below:

- t: Time variable, particularly present time
- t_0: The initial time at which observations start
- T: A preassigned terminal time for optimal control
- $x(t)$, $y(t)$: n- and l-vector stochastic processes representing the system states and the observations respectively, where $x \in E^{(n)}$ and $y \in E^{(l)}$
- $u(t)$: An m-dimensional control vector taking its values in a convex compact subset $\mathcal{U} \subset E^{(m)}$
- $w(t)$, $v(t)$: d_1- and d_2-dimensional Brownian motion processes respectively

$C(t)$, $G(t)$, $R(t)$: $n\times m$, $n\times d_1$ and $l\times d_2$ parameter matrices whose components depend on t

$f[t,x(t)]$, $h[t,x(t)]$: n- and l-vector-valued nonlinear functions, respectively

$\hat{x}(t|t)$: Optimal estimate of $x(t)$, i.e. $\hat{x}(t|t)=E\{x(t)|\mathcal{Y}_t\}$

$P(t|t)$: Error covariance matrix in optimal estimate of $x(t)$ conditioned by $\mathcal{Y}_t$, i.e. $P(t|t)=\mathrm{cov.}[x(t)|\mathcal{Y}_t]$.

2.1. Stochastic Integral of Itô-type and Stochastic Differential Equation

Guided by the well-known state space representation concept, the dynamics of an important class of dynamical systems in the field of engineerings can be described by a nonlinear vector differential equation of the following form,

$$\frac{dx(t,\omega)}{dt} = f[t,x(t,\omega)] + c[t,u(t)] + G[t,x(t,\omega)]\gamma(t,\omega), \quad t\varepsilon[t_0,T], \tag{2.1}$$

where $x(t,\omega)$ is an n-vector, the state of the system; $f[t,x(t,\omega)]$ is an n-vector nonlinear function; $c[t,u(t)]$ is an n-vector forcing term; $u(t)$ is an m-vector control signal to be specified in the later chapters; $G[t,x(t,\omega)]$ is an $n\times m$ matrix; and $\gamma(t,\omega)$ is a d_1-vector white Gaussian noise process with zero-mean and covariance matrix

$$E\{\gamma(t,\omega)\gamma'(\tau,\omega)\} = I\delta(t-\tau).$$

Much of the difficulty in the initial work in the area of optimal nonlinear estimation centered around certain ambiguity that arose in the interpretation of Eq.(2.1).* The white Gaussian noise process $\{\gamma(t),\ t\varepsilon[t_0,T]\}$ was introduced as a means of expressing random disturbances. Such a type as Eq.(2.1) is sometimes called a Langevin equation.

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* In the early development of nonlinear filtering, there were differences between results obtained by Kushner[74,75] and by Stratonovich[121]. It was shown that the differences were due to the differences in the interpretation of equations of the type given in Eq.(2.1). An excellent discussion of these differences can be found in Jazwinski [54].

Now the $\{\gamma(t)\}$ process is delta-correlated and its sample functions are delta functions, and as a result, $\gamma(t)$ is neither mean square Riemann integrable, nor is integrable w.p.1. Consequently, (2.1) loses its mathematical meaning. Recalling that white Gaussian noise is the formal derivative of Brownian motion process, let us introduce a d_1-process of independent Brownian motions through the relation,[54,127,157,163]

$$\text{(2.2)} \qquad w(t) = \int^{t} \gamma(s)ds.$$

Once the Brownian motion process has been defined, the formal equation (2.1) can be integrated and replaced by the integral equation,

$$\text{(2.3)} \qquad x(t) = x(t_0) + \int_{t_0}^{t} f[s,x(s)]ds + \int_{t_0}^{t} c[s,u(s)]ds + \int_{t_0}^{t} G[s,x(s)]dw(s).$$

With appropriate restrictions placed on the functions $f[s,x(s)]$ and $c[s,u(s)]$, the first two integrals in the above equation are the ordinary Riemann integrals for the sample functions. Since the Brownian motion process is of unbounded variation, the last integral which is specified as stochastic integral cannot be interpreted in the Lebesgue-Stieltjes sense. In order to give Eq.(2.3) a precise meaning, we must modify the usual definition of the integral. In this section we summarize the basic elements of the Itô theory of the stochastic integral. With this theory Eq.(2.3) can be given a precise interpretation.[28]

(7) *Brownian motion process:* Let $(\Omega,\mathcal{S},P)$ be the basic probability space. Let $\mathcal{S}_s$ be a monotone family of σ-algebras from $\mathcal{S}$. The stochastic process $\{w(t),\ t\varepsilon[t_0,T]\}$ is called a Brownian motion (Wiener) process with respect to $\mathcal{S}_s$, if

(i) $w(t)$ is $\mathcal{S}_t$-measurable for each $t\varepsilon[t_0,T]$

(ii) $w(t)$ is a process with independent increments

(iii) the random variables $w(t)-w(s)$ $(s<t)$ are real-valued and normally distributed with

$$E\{w(t)-w(s)|\mathcal{S}_s\} = 0$$

$$E\{[w(t)-w(s)][w(t)-w(s)]'|S_s\} = I(t-s)$$

(iv) $P\{w(t_0)=0\} = 1.$

(8) *Itô stochastic integral*: Let $\{w(t),\ t\varepsilon[t_0,T]\}$ be a scalar Brownian motion process and let $\phi(t,\omega)$ be a scalar function such that

(i) $\phi(t,\omega)$ is jointly measurable in (t,ω)

(ii) for each t, $\phi(t,\omega)$ is measurable with respect to S_t

(iii) $\int_{t_0}^{T} E\{|\phi(t,\omega)|^2\}dt < \infty$.*

The stochastic integral is defined as

$$\int_{t_0}^{T} \phi(t,\omega)dw(t) = \underset{n\to\infty}{\mathrm{l.i.m.}} \sum_{i=0}^{n-1} \phi(t_i^{(n)},\omega)[w(t_{i+1}^{(n)})-w(t_i^{(n)})], \tag{2.4}$$

where $\lim_{n\to\infty} \max_i (t_{i+1}^{(n)}-t_i^{(n)})=0$.

The definition of the scalar Itô integral can be easily generalized to the vector case.

Now the third integral in (2.3) is well defined as (7), and therefore Eq.(2.3) can well be interpreted in a meaningful way.

In the remainder of this section, the principal concepts of the Itô theory of stochastic differential equation are presented; this theory is used throughout this dissertation as a model for stochastic dynamical systems.

(9) *Itô process*: Let w(t) be a Brownian motion process. A stochastic process $\{x(t),\ t\varepsilon[t_0,T]\}$ is called an Itô process with respect to the Brownian motion process w(t), relative to the pair of functions $f(t,\omega)$ and $G(t,\omega)$ if

$$x(t) - x(t_0) = \int_{t_0}^{t} f(s,\omega)ds + \int_{t_0}^{t} G(s,\omega)dw(s). \tag{2.5}$$

From the definition of Itô stochastic integral, the following

* The definition of the stochastic integral can be generalized to the functions which are L^2 a.s., that is, the functions which satisfy, instead of (iii), $P\{\int_{t_0}^{T}|\phi(t,\omega)|^2dt<\infty\}=1$ (see Itô [50], Skorokhod [118] and McKean [91]). The condition (iii) is sufficient for our work.

conditions are sufficient to insure that the right-hand side of (2.5) is well defined and continuous in t.

(A2.1) $f(t,\omega)$ and $G(t,\omega)$ are nonanticipating functions, that is these have properties (i) and (ii) in (8).

(A2.2) $\int_{t_0}^{T} E\{|f(s,\omega)|\}ds<\infty$ and $\int_{t_0}^{T} E\{|G(s,\omega)|^2\}ds<\infty$.

(A2.3) $x(t_0)$ is independent of $w(t)$ for $t>t_0$.

In later, the formal description

$$dx(t) = f(t,\omega)dt + G(t,\omega)dw(t) \tag{2.6}$$

will be used to denote the Itô process (2.5). A special case of practical importance is the Itô process with

$$f(t,\omega) = f[t,x(t,\omega)]$$

and

$$G(t,\omega) = G[t,x(t,\omega)].$$

(10) *Diffusion process*: Let $w(t)$ be a Brownian motion process. A vector Itô process $\{x(t),\ t\varepsilon[t_0,T]\}$ is called the diffusion process with respect to the Brownian motion process $w(t)$ relative to the drift vector $f[t,x(t)]$ and the diffusion matrix $G[t,x(t)]$ if

$$dx(t) = f[t,x(t)]dt + G[t,x(t)]dw(t) \tag{2.7}$$

$$x(t_0) = x_0$$

where

(A2.4) The process $\{w(t),\ t\varepsilon[t_0,T]\}$ is a Brownian motion process of dimension d_1.

(A2.5) $x(t_0)$ is a random variable independent of $\{w(t),\ t\varepsilon[t_0,T]\}$, and $E\{\|x(t_0)\|^2\}<\infty$.

(A2.6) Component of the drift and the diffusion vectors $f(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are Baire functions with respect to the pair (t,ξ) for $t\varepsilon[t_0,T]$ and $-\infty<\xi<\infty$, where $x(t)=\xi$.

(A2.7) (Growth restriction) There exists a positive constant k_1, independent of ξ, such that,

$$\|f(t,\xi)\| \le k_1(1+\|\xi\|^2)^{\frac{1}{2}}$$

$$\|G(t,\xi)\| \leq k_2(1+\|\xi\|^2)^{\frac{1}{2}}.$$

(A2.8) (Lipschitz condition) $f(\cdot,\cdot)$ and $G(\cdot,\cdot)$ satisfy a uniform Lipschitz condition in ξ, that is

$$\|f(t,\xi_1)-f(t,\xi_2)\| \leq k_2\|\xi_1-\xi_2\|$$

$$\|G(t,\xi_1)-G(t,\xi_2)\| \leq k_2\|\xi_1-\xi_2\|.$$

(A2.9) The functions $f(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are uniformly Hölder continuous in t.

Equation (2.7) with assumptions (A2.4)-(A2.9) are referred to as the diffusion process.*

Proposition 2.1. Let $\{x(t),\ t\varepsilon[t_0,T]\}$ be the diffusion process of (2.7). Then $\{x(t)\}$ has the following properties:

(i) for each t in $[t_0,T]$, $x(t)$ is S_t-measurable

(ii) $\int_{t_0}^{T} E\{\|x(t)\|^2\}dt$

(iii) $x(t)$ is sample continuous w.p.1

(iv) the process is uniquely determined by $x(t_0)$ w.p.1

(v) $x(t)$ is a Markov process.

This proposition will be important in this dissertation for making sure the stochastic differential equations which model the dynamics of the systems.

In the following chapters, an extensive use is made of the notion of the Itô differential of an Itô process.

(11) *Itô's differential rule:* Let $x(t)$ be the unique solution of the n-vector Itô stochastic differential equation (2.7). Let $\phi(t,x)$ be a scalar-valued real function, continuously differentiable in t and twice continuously differentiable in x. Then the (stochastic) differential $d\phi$ of ϕ is

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* More strictly speaking, a diffusion process is a strong Markov process with continuous sample paths[77,p.4]. The assumptions (A2.4)-(A2.9) guarantee that $\{x(t),\ t\varepsilon[t_0,T]\}$ is a diffusion process[77,p.15].

$$(2.8)\qquad d\phi = [\frac{\partial\phi}{\partial t}+f'\frac{\partial\phi}{\partial x}+\frac{1}{2}\,\mathrm{tr.}\{G'\frac{\partial^2\phi}{\partial x^2}G\}]dt + (\frac{\partial\phi}{\partial x})'Gdw,$$

where $\partial(\cdot)/\partial x$ denotes the gradient (column) vector and $\partial^2(\cdot)/\partial x^2$ denotes the Hessian matrix of cross partials.

(12) *Itô-Dynkin's formula:* [31,vol.1,p.133] Given the diffusion process (2.7) and let $z(t,x)$ be a real twice continuously differentiable scalar function. Then the conditional expectation of z conditioned on x_0 satisfies

$$(2.9)\qquad E_{x_0}\{z(t,x)\} - z(t_0,x_0) = E_{x_0}\{\int_{t_0}^{t} Lz(s,x)ds\},$$

where L is the differential generator,

$$(2.10)\qquad L(\cdot) = \frac{\partial}{\partial t}(\cdot) + f'(t,x)\frac{\partial}{\partial x}(\cdot) + \frac{1}{2}\,\mathrm{tr.}\{G'(t,x)\frac{\partial^2}{\partial x^2}(\cdot)G(t,x)\}.$$

In this section a brief summary has been given of the Itô theory of stochastic differential equations and this will be one of the main analytical tools for deriving representations for both the optimal estimation and the optimal control problems.

2.2. Alternative Stochastic Differential Equation

In the previous section, the dynamical system equation (2.1) is represented by the precise version of the Itô sense as (2.7) where the forcing term $c[t,u(t)]$ is dropped out. It is well-known that there is another type of versions to Eq.(2.1); i.e. if the stochastic equation (2.1) is interpreted in the sense of Stratonovich, then the equivalent Itô equation is represented by

$$(2.11)\qquad dx_i(t) = [f_i(t,x) + \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{d_1}[G(t,x)]_{kj}\frac{\partial}{\partial x_k}[G(t,x)]_{ij}]dt + \sum_{j=1}^{d_1}[G(t,x)]_{ij}dw_j(t).$$

The Stratonovich-type stochastic integral is "symmetrically" defined by

$$\text{(2.12)}\qquad \int_{t_0}^{T}\phi(t,w(t))dw(t) = \underset{n\to\infty}{\text{l.i.m.}}\sum_{i=0}^{n-1}\phi\left(\frac{t_i^{(n)}+t_{i+1}^{(n)}}{2},\frac{w(t_{i+1}^{(n)})-w(t_i^{(n)})}{2}\right)$$
$$\times[w(t_{i+1}^{(n)})-w(t_i^{(n)})],$$

where $\lim_{n\to\infty}\max_i (t_{i+1}^{(n)}-t_i^{(n)})=0$. Excellent discussions of the relation between Itô and Stratonovich stochastic integrals are found in [54,Chap.4]. It is obvious that the difference between (2.7) and (2.11) is the existence of the term in (2.11),

$$\text{(2.13)}\qquad \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{d_1}[G(t,x)]_{kj}\frac{\partial}{\partial x_k}[G(t,x)]_{ij}.$$

Such a model of (2.11) is used in Chap.5, Sec.5.3 for deriving the filter equation of stochastic system with state-dependent noise.

2.3. Mathematical Models of Dynamical System and Observation Mechanism

As the models given by (1.1) and (1.2) are formal because of the white Gaussian noises, the following couple of stochastic differential equations of the Itô-type are introduced as the precise mathematical models for the system and the observation, based on the rigorous mathematical background of the Itô theory reviewed in Sec.2.1:

$$\text{(2.14)}\qquad dx(t) = f[t,x(t)]dt + c[t,u(t)]dt + G[t,x(t)]dw(t),$$
$$x(t_0) = x_0$$
$$\text{(2.15)}\qquad dy(t) = h[t,x(t)]dt + R[t,x(t)]dv(t),$$
$$y(t_0) = 0.$$

In this section, several types of the models for the dynamical system and the observation process which are used in Part One are defined.

Definition 2.1. (System Σ_0) Let the dynamical system and the observation processes satisfy respectively the stochastic differential equations (2.14) and (2.15). The processes $x(t)$ and $y(t)$ are n- and l-dimensional vector processes respectively ($n\geq l$). In (2.14) and (2.15), the following assumptions are made:

(C0.1) The component of the functions $f[\cdot,\cdot]$, $h[\cdot,\cdot]$, $G[\cdot,\cdot]$ and $R[\cdot,\cdot]$ are Baire functions[28] with respect to the pair (t,ξ) for $t_0 \le t \le T$ and $-\infty<\xi<\infty$, where $x(t)=\xi$.

(C0.2) The functions $f[\cdot,\cdot]$, $h[\cdot,\cdot]$, $G[\cdot,\cdot]$ and $R[\cdot,\cdot]$ satisfy a uniform Lipschitz condition and a growth restriction in the variable ξ.

(C0.3) The functions $f[\cdot,\cdot]$, $h[\cdot,\cdot]$, $G[\cdot,\cdot]$ and $R[\cdot,\cdot]$ are uniformly Hölder continuous in t.

(C0.4) The processes $w(t)$ and $v(t)$ are independent Brownian motion processes of dimensions d_1 and d_2 respectively.

(C0.5) $x(t_0)$ is a random variable independent of both $w(t)$- and $v(t)$-processes.

Equations (2.14) and (2.15) with assumptions (C0.1)-(C0.5) are referred to collectively as the system equations Σ_0.

The control term $c[t,u(t)]$ in (2.14) is specified later in Sec.6.2, defining the class of admissible controls.

Some other systems which are used in the nonlinear filtering problems are defined by slightly modifying the system model Σ_0.

Definition 2.2. (System Σ_{1F}) Let $x(t)$ and $y(t)$ be n-vector dynamical system and l-vector observation processes represented by

$$(2.16)\qquad dx(t) = f[t,x(t)]dt + G(t)dw(t), \quad x(t_0) = x_0$$

$$(2.17)\qquad dy(t) = h[t,x(t)]dt + R(t)dv(t), \quad y(t_0) = 0,$$

where the assumptions (C0.4) and (C0.5) are made and

(C1.1) the nonlinear functions $f[\cdot,\cdot]$ and $h[\cdot,\cdot]$ are Baire functions with respect to the pair (t,ξ), and satisfy a uniform Lipschitz condition and a growth restriction in the variable ξ and are uniformly Hölder continuous in t,

(C1.2) the parameter matrices $G(t)$ and $R(t)$ are $n\times d_1$- and $l\times d_2$-dimensional, measurable and bounded on the finite time interval $[t_0,T]$,

(C1.3) $\{R(t)R'(t)\}$ is nonsingular and is bounded on $[t_0,T]$.

Equations (2.16) and (2.17) are collectively specified as Σ_{1F}.

Definition 2.3. (System Σ_{2F}) Let $x(t)$ and $y(t)$ be n- and l-vector stochastic processes represented by

(2.18) $$dx(t) = f[t,x(t)]dt + G_0(t)dw_1(t) + G[t,x(t)]dw_2(t)$$

$$x(t_0) = x_0$$

(2.19) $$dy(t) = h[t,x(t)]dt + R(t)dv(t)$$

$$y(t_0) = 0,$$

where the assumption (C1.1) in Def.2.2 is made and

(C2.1) $w_1(t)$, $w_2(t)$ and $v(t)$ are mutually independent d_1-, d_2- and d_3-vector Brownian motion processes,

(C2.2) $x(t_0)$ is independent of the Brownian motion processes,

(C2.3) $G_0(t)$ and $R(t)$ are $n\times d_1$- and $l\times d_3$-matrices which are measurable and bounded in t, and $\{R(t)R'(t)\}$ is nonsingular,

(C2.4) $G[t,x(t)]$ is given by

$$G[t,x(t)] = \sum_{i=1}^{n} G_i(t)x_i$$

where the $G_i(t)$ are continuous bounded matrix-valued functions of t with dimension $n\times d_2$.

Equations (2.18) and (2.19) with (C2.1)-(C2.4) are specified as Σ_{2F}.

Further the following system Σ_{3F} is defined.

Definition 2.4. (System Σ_{3F}) Let $x(t)$ and $y(t)$ be n- and l-vector processes represented by

(2.20) $$dx(t) = f[t,x(t)]dt + G_0(t)dw_1(t) + dW_2(t)x(t)$$

$$x(t_0) = x_0$$

(2.21) $$dy(t) = h[t,x(t)]dt + R_0(t)dv_1(t) + dV_2(t)r[t,x(t)]$$

$$y(t_0) = 0,$$

where (C1.1), (C2.2) are made and

(C3.1) $w_1(t)$, $v_1(t)$, $W_2(t)$ and $V_2(t)$ are mutually independent d_1- d_2-vector and $n\times n$-, $l\times l$-matrix Brownian motion processes with zero mean, and

$$E\{dw_1(t)dw_1'(t)\} = Idt$$

$$E\{dv_1(t)dv_1'(t)\} = Idt$$

$$E\{(dw_{2ij})(dw_{2kl})\} = \begin{cases} \phi_{ij}dt & \text{for } i=k \text{ and } j=l \\ 0 & \text{for } i\neq k \text{ or } j\neq l \end{cases}$$

$$E\{(dv_{2ij})(dv_{2kl})\} = \begin{cases} \lambda_{ij}dt & \text{for } i=k \text{ and } j=l \\ 0 & \text{for } i\neq k \text{ or } j\neq l \end{cases}$$

where ϕ_{ij} and λ_{ij} are the (i,j)-elements of the matrices Φ and Λ respectively,

(C3.2) $r(t,x)$ is an n-vector-valued Baire function which satisfies a uniform Lipschitz and a growth restriction conditions.

(C3.3) $G_0(t)$ and $R_0(t)$ are $n\times d_1$- and $l\times d_2$-matrices and $\{R_0(t)R_0'(t)\}$ is nonsingular.

Equations (2.20) and (2.21) with (C3.1)-(C3.3) are specified as Σ_{3F}.

The systems Σ_{iC} (i=1,2,3) are defined which correspond to the above defined systems Σ_{iF} as follows.

Definition 2.5. (Systems Σ_{iC}) The systems Σ_{iC} for i=1,2,3 are specified by adding the control term $c[t,u(t)]dt$ in the right-hand side of (2.16), (2.18) and (2.20) respectively such as, for instance,

$$\Sigma_{1C}: \begin{cases} dx(t) = f[t,x(t)]dt + c[t,u(t)]dt + G(t)dw(t) \\ dy(t) = h[t,x(t)]dt + R(t)dv(t). \end{cases}$$

CHAPTER 3. REVIEW OF STOCHASTIC LINEARIZATION IN MARKOVIAN FRAMEWORK

3.1. Introductory Remarks

In the nonlinear filtering and control theory, the approximation of the nonlinear function by some linear one will play an important role as might be expected. Limiting discussions to the filtering theory, several approximation techniques are presented as stated in Sec.1.1.B. A familiar technique is the expansion of the nonlinear function into a Taylor series up to the suitable order terms. Such a technique was used by Schwartz[111]. However, anoter powerful technique was suggested by Sunahara[126], and the filtering problem was solved.

The author reviews briefly the stochastic linearization technique in the following sections in order to use such linearization technique for realizing an overall configuration of the optimal nonlinear control system subjected to the observation noise.

3.2. Stochastic Linearization in Markovian Framework[126]

The system function $f[t,x(t)]$ in (2.16) is expanded into

$$f[t,x(t)] = a(t) + B(t)\{x(t)-\hat{x}(t|t)\} + \varepsilon_f(t), \tag{3.1}$$

where $\varepsilon_f(t)$ denotes the collection of n-dimensional vector error terms and $a(t)$, $B(t)$ are an n-dimensional vector and n×n matrix, respectively. The linearization coefficients $a(t)$ and $B(t)$ are determined in such a way that the conditional expectation of the squared norm of $\varepsilon_f(t)$,

$$E\{\|\varepsilon_f(t)\|^2|\mathcal{Y}_t\} = E\{\|f[t,x(t)]-a(t)-B(t)\{x(t)-\hat{x}(t|t)\}\|^2|\mathcal{Y}_t\}, \tag{3.2}$$

becomes minimal. The necessary and sufficient conditions to minimize (3.2) are

$$a(t) = E\{f[t,x(t)]|\mathcal{Y}_t\} \triangleq \hat{f}[t,x(t)] \tag{3.3a}$$

and

$$B(t) = E\{[f[t,x(t)]-\hat{f}[t,x(t)]][x(t)-\hat{x}(t|t)]'|\mathcal{Y}_t\}P^{-1}(t|t), \tag{3.3b}$$

where

$$P(t|t) = \mathrm{cov.}[x(t)|\mathcal{Y}_t]. \tag{3.4}$$

In evaluating $a(t)$ and $B(t)$, we have two problems at hand. One is to compute the state estimate $\hat{x}(t|t)$ and the error covariance $P(t|t)$ and the other is to evaluate the conditional expectation $E\{\cdot|\mathcal{Y}_t\}$. For evaluating the conditional probability density function $p\{x(t)|\mathcal{Y}_t\}$, this is assumed to be Gaussian with the mean value $\hat{x}(t|t)$ and the covariance matrix $P(t|t)$, i.e.

$$p\{x(t)|\mathcal{Y}_t\} = (2\pi)^{-\frac{n}{2}}|P(t|t)|^{-\frac{1}{2}} \times\exp\{-\frac{1}{2}\|x(t)-\hat{x}(t|t)\|^2_{P^{-1}(t|t)}\}. \tag{3.5}$$

With the help of this Gaussian assumption, both $a(t)$ and $B(t)$ can be obtained in the form, $a(t)=a(t,\hat{x}(t|t),P(t|t))$ and $B(t)=B(t,\hat{x}(t|t),P(t|t))$... Futhermore, the (i,j)-th element of the matrix $B(t)$ is simply obtained by

$$(3.6) \qquad b_{ij}(t) = \frac{\partial a_i(t)}{\partial \hat{x}_j(t|t)}.$$

A striking fact is that the random variables $a(t)$ and $B(t)$ are not independent but dependent mutually on the state estimate $\hat{x}(t|t)$ and the error covariance matrix $P(t|t)$. From this point of view, more precise symbols, $a(t,\hat{x}(t|t),P(t|t))$ and $B(t,\hat{x}(t|t),P(t|t))$ should be introduced. However, for economy of descriptions, we merely denote these by $a(t)$ and $B(t)$ without indicating the dependence on both $\hat{x}(t|t)$ and $P(t|t)$.

Using $a(t)$ and $B(t)$ obtained in (3.3a) and (3.3b), the nonlinear function $f[t,x(t)]$ is replaced by the quasi-linear function, $a(t)+B(t)\{x(t)-\hat{x}(t|t)\}$, and then the nonlinear differential equation (2.16) is approximated by

$$(3.7) \qquad dx(t) = B(t)x(t)dt + \{a(t)-B(t)\hat{x}(t|t)\}dt + G(t)dw(t).$$

In the following analysis of this dissertation, the stochastic linearization technique just reviewed shows to be very attractive and plays an important role.

3.3. Error Evaluation of the Stochastic Linearization

In order to evaluate the stochastic linearization, let us consider the following n-dimensional stochastic differential equation,

$$(3.8) \qquad dx(t) = f[t,x(t)]dt + G(t)dw(t), \quad t_0 \le t \le T.$$

In (3.8), the state $x(t)$ is completely observable and the nonlinear function $f(t,x)$ satisfies a uniform Lipschitz condition and is uniformly bounded, (see Sec.2.1, Chap.2)

$$(A3.1) \qquad \|f(t,x)-f(t,z)\| \le c\|x-z\|$$

$$(A3.2) \qquad \|f(t,x)\| \le c_0(1+x'x)^{\frac{1}{2}},$$

where, in (A3.1) and (A3.2), $x,z \in E^{(n)}$ and c, c_0 are real positive constants and independent of both t and x.

A precise interpretation of (3.8) is

$$(3.9) \qquad x(t) = x(t_0) + \int_{t_0}^{t} f[s,x(s)]ds + \int_{t_0}^{t} G(s)dw(s).$$

In the sequel, the solution of (3.9) is written as $x^o(t)$ in order to discriminate it from the quasi-linearized solution $x_a(t)$ which is generated by the quasi-linearized stochastic differential equation described later.

The stochastic linearization technique reviewed in the previous section is modified where the state variable is completely observable as follows. Expand the function $f[t,x^o(t)]$ into

$$(3.10)\qquad f[t,x^o(t)] = a(t) + B(t)\{x^o(t)-x_a(t)\} + \varepsilon_f(t),$$

where $a(t)$ and $B(t)$ are determined under the criterion, $\min_{a(t),B(t)} E\{\|\varepsilon_f(t)\|^2 | x^o(t_0)=x_0\}$, as

$$(3.11a)\qquad a(t) = E\{f[t,x^o(t)] | x^o(t_0)=x_0\} \triangleq \bar{f}[t,x^o(t)]$$

$$(3.11b)\qquad B(t) = E\{[f[t,x^o(t)]-\bar{f}[t,x^o(t)]][x^o(t)-\bar{x}_a(t)]' | x^o(t_0)=x_0\} \times P^{-1}(t),$$

where

$$(3.12)\qquad P(t) = \mathrm{cov.}[x^o(t) | x^o(t_0)=x_0].$$

Then the sample path $x^o(t)$ determined by (3.8) is approximated by

$$(3.13)\qquad dx_a(t) = B(t)x_a(t)dt + \{a(t)-B(t)\bar{x}_a(t)\}dt + G(t)dw(t),$$

whose interpretation is given by

$$(3.14)\qquad x_a(t) = x_a(t_0) + \int_{t_0}^{t} [a(s)+B(s)\{x_a(s)-\bar{x}_a(s)\}]ds + \int_{t_0}^{t} G(s)dw(s).$$

In (3.10) to (3.14), $\bar{x}_a(t)$ is a solution of the differential equation

$$(3.15)\qquad \frac{d\bar{x}_a(t)}{dt} = \bar{f}[t,x^o(t)], \quad \bar{x}_a(t_0) = E\{x(t_0)\}.$$

We evaluate the expected squared error,

$$(3.16)\qquad E_{x_0}\{\|x^o(t)-x_a(t)\|^2\},$$

where $E_{x_0}\{\cdot\}$ denotes the conditional expectation conditioned by $x(t_0)=x_0$ In the evaluation of (3.16), the following assumption and lemmas are needed:

(A3.3) The parameter matrix $G(t)$ is bounded; that is, there exists a constant γ such that

$$\max_{t_0 \leq t \leq T} \|G(t)\| \leq \gamma.$$

Lemma 3.1. Assume (A3.2). Then there exists a nonnegative constant δ such that

$$\max_{t_0 \leq t \leq T} E_{x_0}\{\|f(t,x)-\bar{f}(t,x)\|^2\} \leq \delta^2.$$

Proof. Note that

$$\begin{aligned} (3.17) \qquad \|\bar{f}(t,x)\|^2 &= \|\int_{E^{(n)}} f(t,x)p\{t,x|x_0\}dx\|^2 \\ &\leq \int_{E^{(n)}} \|f(t,x)\|^2 p\{t,x|x_0\}dx \\ &\leq \int_{E^{(n)}} c_0^2(1+x'x)p\{t,x|x_0\}dx \\ &= c_0^2[1+E_{x_0}\{x'x\}], \end{aligned}$$

where (A3.2) was used. Hence by (A3.2) and (3.17), we have

$$\begin{aligned} E_{x_0}\{\|f(t,x)-\bar{f}(t,x)\|^2\} &\leq 2E_{x_0}\{\|f(t,x)\|^2\} + 2E_{x_0}\{\|\bar{f}(t,x)\|^2\} \\ &\leq 2E_{x_0}\{c_0^2(1+x'x)\} + 2E_{x_0}\{c_0^2[1+E_{x_0}\{x'x\}]\} \\ &= 4c_0^2[1+E_{x_0}\{x'x\}], \end{aligned}$$

which shows that there exists a constant δ such that*

$$(3.18) \qquad \max_{t_0 \leq t \leq T} E_{x_0}\{\|f(t,x)-\bar{f}(t,x)\|^2\} \leq \delta^2 < \infty.$$

(Q.E.D.)

Lemma 3.2. The linearization coefficients $a(t)$ and $B(t)$ are bounded; i.e.,

* Actually, $E_{x_0}\{\|x\|^2\}<M$ (const.) on $[t_0,T]$. See [54, p.106].

there exist some constants α and β such that

$$\max_{t_0 \le t \le T} \|a(t)\|^2 \le \alpha^2, \qquad \max_{t_0 \le t \le T} \|B(t)\|^2 \le \beta^2.$$

Proof. The fisrt boundedness of $a(t)$ is obvious since from (3.17) an the footnote on p.29;

$$\max_{t_0 \le t \le T} \|a(t)\|^2 = \max_{t_0 \le t \le T} \|\bar{f}(t,x^o)\|^2$$

$$\le c_0^2[1 + \max_{t_0 \le t \le T} E_{x_0}\{\|x^o\|^2\}] \le \alpha^2.$$

Next, by (3.11b)

$$\|B(t)\|^2 = \|E_{x_0}\{[f(t,x^o)-\bar{f}(t,x^o)](x^o-\bar{x}_a)'\}P^{-1}(t)\|^2 \tag{3.19}$$

$$\le \|E_{x_0}\{[f(t,x^o)-\bar{f}(t,x^o)](x^o-\bar{x}_a)'\}\|^2\|P^{-1}(t)\|^2$$

$$\le E_{x_0}\{\|f(t,x^o)-\bar{f}(t,x^o)\|^2\}E_{x_0}\{\|x^o-\bar{x}_a\|^2\}\|P^{-1}(t)\|^2$$

$$\le \delta^2 E_{x_0}\{\|x^o-\bar{x}_a\|^2\}\|P^{-1}(t)\|^2,$$

where the Cauchy-Buniakovskii inequality and Lemma 3.1 were used.

Now evaluate $E_{x_0}\{\|x^o-\bar{x}_a\|^2\}$ in (3.19). From (3.9) and by (3.15),

$$E_{x_0}\{\|x^o(t)-\bar{x}_a(t)\|^2\} \tag{3.20}$$

$$= E_{x_0}\{\|x^o(t_0) - \bar{x}_a(t_0) + \int_{t_0}^{t} [f(s,x^o)-\bar{f}(s,x^o)]ds$$

$$+ \int_{t_0}^{t} G(s)dw(s)\|^2\}$$

$$\le 2E_{x_0}\{\|x^o(t_0)-\bar{x}_a(t_0)\|^2\}$$

$$+ 2E_{x_0}\{\|\int_{t_0}^{t} [f(s,x^o)-\bar{f}(s,x^o)]ds + \int_{t_0}^{t} G(s)dw(s)\|^2\},$$

where the relation $(x+y)^2 \le 2x^2+2y^2$ was used. Here,

$$E_{x_0}\{\|x^o(t_0)-\bar{x}_a(t_0)\|^2\} = \text{tr.}\{\text{cov.}[x^o(t_0)]\} \tag{3.21}$$

$$= \text{tr. } P(t_0)$$

and

$$(3.22)\qquad E_{x_0}\{\|\int_{t_0}^{t}[f(s,x^o)-\bar{f}(s,x^o)]ds + \int_{t_0}^{t}G(s)dw(s)\|^2\}$$

$$\leq 2E_{x_0}\{\int_{t_0}^{t}\|f(s,x^o)-\bar{f}(s,x^o)\|^2 ds\} + 2E_{x_0}\{\|\int_{t_0}^{t}G(s)dw(s)\|^2\}.$$

In (3.22), note that by Lemma 3.1 and (A3.3)

$$(3.23)\qquad E_{x_0}\{\int_{t_0}^{t}\|f(s,x^o)-\bar{f}(s,x^o)\|^2 ds\} \leq \delta^2(t-t_0)$$

and

$$(3.24)\qquad E_{x_0}\{\|\int_{t_0}^{t}G(s)dw(s)\|^2\} = \int_{t_0}^{t}\|G(s)\|^2 ds \leq \gamma^2(t-t_0).$$

Then, combining (3.20)-(3.24) and rearranging terms, we have

$$(3.25)\qquad E_{x_0}\{\|x^o(t)-\bar{x}_a(t)\|^2\} \leq 2\mathrm{tr}.P(t_0) + 4(\delta^2+\gamma^2)(t-t_0).$$

Hence from (3.19) and (3.25),

$$(3.26)\qquad \|B(t)\|^2 \leq 2\delta^2[\mathrm{tr}.P(t_0)+2(\delta^2+\gamma^2)(t-t_0)]\|P^{-1}(t)\|^2.$$

From (3.26) it is obvious that there exists a constant β such that

$$(3.27)\qquad \max_{t_0\leq t\leq T}\|B(t)\|^2 \leq \beta^2.$$

(Q.E.D.)

With hypotheses (A3.1)-(A3.3) and Lemmas 3.1 and 3.2, we have the following theorem.

Theorem 3.1. Suppose that the hypotheses (A3.1)-(A3.3) hold. Then

$$(3.28)\qquad E_{x_0}\{\|x^o(t)-x_a(t)\|^2\} \leq (t-t_0)q_t$$

and

$$(3.29)\qquad P_{x_0}\{\sup_{t_0\leq s\leq t}\|x^o(s)-x_a(s)\|>\varepsilon\} \leq \frac{1}{\varepsilon^2}(t-t_0)q_t,$$

where $P_{x_0}\{\cdot\}$ denotes the conditional probability given x_0, and

$$(3.30) \qquad q_t \triangleq 2(\delta^2 - \frac{\gamma^2}{T-t_0})(t-t_0)$$

$$+ \frac{1}{2(T-t_0)} [2\mathrm{tr}.P(t_0) + \frac{\gamma^2}{\beta^2(T-t_0)}][e^{4\beta^2(T-t_0)(t-t_0)} - 1].$$

Proof. The proof of the theorem is straightforward. Viewing (3.9) and (3.13) and noting $x(t_0)=x_a(t_0)=x_0$, it follows that

$$(3.31) \qquad E_{x_0}\{\|x^o(t)-x_a(t)\|^2\}$$

$$= E_{x_0}\{\|\int_{t_0}^{t} [f(s,x^o(s))-[a(s)+B(s)\{x_a(s)-\bar{x}_a(s)\}]]ds\|^2\}$$

$$\leq E_{x_0}\{[\int_{t_0}^{t} \|f(s,x^o(s))-[a(s)+B(s)\{x_a(s)-\bar{x}_a(s)\}]\| ds]^2\}$$

$$\leq E_{x_0}\{[\int_{t_0}^{t} [\|f(s,x^o(s))-\bar{f}(s,x^o(s))\|+\|B(s)\|\,\|x_a(s)-\bar{x}_a(s)\|]ds]^2\}$$

$$\leq (t-t_0)E_{x_0}\{\int_{t_0}^{t} [\|f(s,x^o(s))-\bar{f}(s,x^o(s))\|+\|B(s)\|\,\|x_a(s)-\bar{x}_a(s)\|]^2 d$$

$$\leq 2(t-t_0)[E_{x_0}\{\int_{t_0}^{t} \|f(s,x^o(s))-\bar{f}(s,x^o(s))\|^2 ds\}$$

$$+ \int_{t_0}^{t} E_{x_0}\{\|B(s)\|^2\|x_a(s)-\bar{x}_a(s)\|^2\}ds],$$

where the Cauchy-Buniakovskii inequality was used. Now, by Lemma 3.1 an (A3.3), the relation (3.23) also holds; and by Lemma 3.2 the second integrand of the right-hand side of (3.31) is evaluated as

$$(3.32) \qquad E_{x_0}\{\|B(s)\|^2\|x_a(s)-\bar{x}_a(s)\|^2\} \leq \beta^2 E_{x_0}\{\|x_a(s)-\bar{x}_a(s)\|^2\}.$$

Let us tern our eyes to evaluate $E_{x_0}\{\|x_a(s)-\bar{x}_a(s)\|^2\}$. A similar method to Lemma 3.2 is applied. From (3.14) and (3.15), we have

$$(3.33) \qquad E_{x_0}\{\|x_a(s)-\bar{x}_a(s)\|^2\} = E_{x_0}\{\|x_a(t_0)-\bar{x}_a(t_0)$$

$$+ \int_{t_0}^{s} B(\tau)\{x_a(\tau)-\bar{x}_a(\tau)\}d\tau + \int_{t_0}^{s} G(\tau)dw(\tau)\|^2\}$$

$$\leq 2E_{x_0}\{\|x_a(t_0)-\bar{x}_a(t_0)\|^2\} + 2E_{x_0}\{\|\int_{t_0}^{s} B(\tau)\{x_a(\tau)-\bar{x}_a(\tau)\}d\tau+\int_{t_0}^{s} G(\tau)dw(\tau$$

Here,

$$(3.34)\qquad E_{x_0}\{\|x_a(t_0)-\bar{x}_a(t_0)\|^2\} = \text{tr.}\{\text{cov.}[x_a(t_0)|x_0]\}$$
$$\equiv \text{tr.}\{\text{cov.}[x^o(t_0)]\} = \text{tr.}P(t_0)$$

and

$$(3.35)\qquad E_{x_0}\{\|\int_{t_0}^{s} B(\tau)\{x_a(\tau)-\bar{x}_a(\tau)\}d\tau + \int_{t_0}^{s} G(\tau)\,dw(\tau)\|^2\}$$
$$\leq 2E_{x_0}\{[\int_{t_0}^{s}\|B(\tau)\{x_a(\tau)-\bar{x}_a(\tau)\}\|\,d\tau]^2\}$$
$$+ 2E_{x_0}\{\|\int_{t_0}^{s} G(\tau)\,dw(\tau)\|^2\}$$
$$\leq 2(s-t_0)E_{x_0}\{\int_{t_0}^{s}\|B(\tau)\|^2\|x_a(\tau)-\bar{x}_a(\tau)\|^2 d\tau\}$$
$$+ 2E_{x_0}\{\|\int_{t_0}^{s} G(\tau)\,dw(\tau)\|^2\}$$
$$\leq 2\beta^2(s-t_0)\int_{t_0}^{s} E_{x_0}\{\|x_a(\tau)-\bar{x}_a(\tau)\|^2\}d\tau + 2\gamma^2(s-t_0).$$

In (3.35), Lemma 3.2 and (A3.3) were used. Then combination of (3.33)-(3.35) and rearrangement of terms yield

$$(3.36)\qquad E_{x_0}\{\|x_a(s)-\bar{x}_a(s)\|^2\} \leq 2\text{tr.}P(t_0) + 4\gamma^2(s-t_0)$$
$$+ 4\beta^2(s-t_0)\int_{t_0}^{s} E_{x_0}\{\|x_a(\tau)-\bar{x}_a(\tau)\|^2\}d\tau$$
$$\leq 2\text{tr.}P(t_0) + 4\gamma^2(s-t_0) + 4\beta^2(T-t_0)\int_{t_0}^{s} E_{x_0}\{\|x_a(\tau)-\bar{x}_a(\tau)\|^2\}d\tau.$$

We need the following lemma.

Lemma 3.3. (Gronwall-Bellman Lemma[21; 44,p.393]) Let $\alpha(t)$ denote a nonnegative integrable function that is defined for $t\varepsilon[t_0,T]$ and that satisfies the inequality

$$(3.37)\qquad \alpha(t) \leq \beta(t) + k\int_{t_0}^{t} \alpha(s)\,ds,$$

where k is a nonnegative constant and $\beta(t)$ is an integrable function.

Then

$$(3.38) \qquad \alpha(t) \leq \beta(t) + k\int_{t_0}^{t} e^{k(t-s)}\beta(s)\,ds.$$

Applying Lemma 3.3 to (3.36), we have

$$(3.39) \qquad E_{x_0}\{\|x_a(s)-\bar{x}_a(s)\|^2\} \leq 2tr.P(t_0) + 4\gamma^2(s-t_0)$$

$$+ 8\beta^2(T-t_0)\int_{t_0}^{s} e^{4\beta^2(T-t_0)(s-\tau)}[tr.P(t_0)+2\gamma^2(\tau-t_0)]d\tau$$

$$= [2tr.P(t_0) + \frac{\gamma^2}{\beta^2(T-t_0)}]e^{4\beta^2(T-t_0)(s-t_0)} - \frac{\gamma^2}{\beta^2(T-t_0)}.$$

Therefore, combining (3.23), (3.32),(3.39) with (3.31) and performing the integration, we have the result (3.28).

In the followings, let us evaluate the probability, $P_{x_0}\{\sup_{t_0\leq s\leq t}\|x^o(s)-x_a(s)\|>\varepsilon\}$. In view of (3.31), we have

$$(3.40) \qquad P_{x_0}\{\sup_{t_0\leq s\leq t}\|x^o(s)-x_a(s)\|>\varepsilon\}$$

$$\leq P_{x_0}\{\int_{t_0}^{t}\|f(s,x^o(s))-[a(s)+B(s)\{x_a(s)-\bar{x}_a(s)\}]\|ds>\varepsilon\}$$

$$= P_{x_0}\{[\int_{t_0}^{t}\|f(s,x^o(s))-[a(s)+B(s)\{x_a(s)-\bar{x}_a(s)\}]\|ds]^2>\varepsilon^2\}.$$

By using the Chebychev inequality and further the Cauchy-Buniakovskii inequality, it follows that

$$P_{x_0}\{\sup_{t_0\leq s\leq t}\|x^o(s)-x_a(s)\|>\varepsilon\}$$

$$\leq \frac{1}{\varepsilon^2}E_{x_0}\{[\int_{t_0}^{t}\|f(s,x^o(s))-[a(s)+B(s)\{x_a(s)-\bar{x}_a(s)\}]\|ds]^2\}$$

$$\leq \frac{1}{\varepsilon^2}(t-t_0)E_{x_0}\{\int_{t_0}^{t}\|f(s,x^o(s))-[a(s)+B(s)\{x_a(s)-\bar{x}_a(s)\}]\|^2ds\}$$

$$\leq \frac{2}{\varepsilon^2}(t-t_0)[\delta^2(t-t_0) + \beta^2\int_{t_0}^{t}E_{x_0}\{\|x_a(s)-\bar{x}_a(s)\|^2\}ds].$$

Substitution of (3.39) into (3.41) yields (3.29). This completes the proof.

3.4. Relations between Stochastic Linearization and Classical Statistical Equivalent Linearization

Although the stochastic linearization technique reviewed in Sec.3.2 allows us to assume that the additive random noise is nonstationary Gaussian, we shall assume, in this section, the additive noise to be stationary Gaussian in order to examine some relations between the stochastic linearization and the classical statistical equivalent linearization.

Consider an n-dimensional nonlinear dynamical system

$$dx(t) = f(x)dt + Gdw(t), \tag{3.42}$$

where $w(t)$ is a Brownian motion process with covariance $\sigma^2 I$. The quasi-linearized system is given by

$$dx(t) = [a+B\{x(t)-\bar{x}(t)\}]dt + Gdw(t). \tag{3.43}$$

In (3.43), a and B are the linearization-coefficients and $\bar{x}$ denotes the conditional expectation of $x(t)$ conditioned by the initial state $x(t_0)$, i.e. $\bar{x}(t)=E\{x(t)|x(t_0)=x_0\}$. The covariance of $x(t)$, $P(t)=E\{(x-\bar{x})(x-\bar{x})'| x(t_0)\}$, satisfies the equation,

$$\frac{dP(t)}{dt} = BP(t) + P(t)B' + \sigma^2 GG'. \tag{3.44}$$

The basic concept of the classical statistical linearization which was examined in detail by Sawaragi and Sunahara[107,108] can be shown in Fig.3.1.

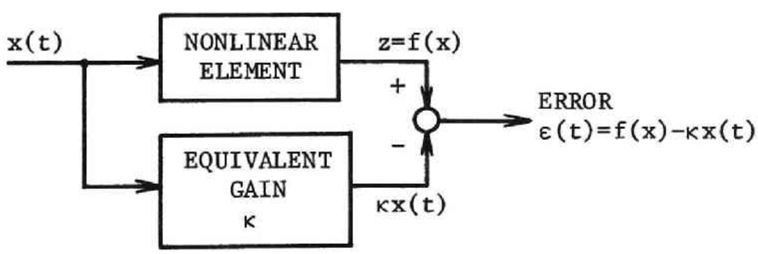

Fig.3.1. Basic concept of statistical linearization.

The output of the nonlinear element $z(t)$ is evaluated by the approximated signal

(3.45) $\quad z(t) = \kappa x(t),$

where κ is known as the ($n\times n$-dimensional matrix) statistical equivalent gain of the nonlinear function $f(x)$. The coefficient κ is determined so as to minimize the criterion

(3.46) $\quad E\{\| f(x)-\kappa x(t)\|^2\}.$

In the case where $x(t)$ is stationary, the gain κ yields to

(3.47) $\quad \kappa = [\int_{E^{(n)}} f(x)x'p(x)dx][\int_{E^{(n)}} xx'p(x)dx]^{-1},$

where $p(x)$ is the stationary probability density function(pdf) of $x(t)$. Equation (3.47) may be represented as

(3.48) $\quad \kappa = E\{f(x)x'\}\Psi_x^{-1},$

where Ψ_x is the covariance of $x(t)$ defined by

(3.49) $\quad \Psi_x \triangleq \int_{E^{(n)}} xx'p(x)dx.$

If the pdf of $x(t)$ is assumed to be Gaussian with zero-mean and the covariance Ψ_x,

(3.50) $\quad p(x) = (2\pi)^{-\frac{n}{2}}|\Psi_x|^{-\frac{1}{2}}\exp\{-\frac{1}{2}x'\Psi_x^{-1}x\},$

the equivalent gain κ becomes a function of the covariance matrix Ψ_x:

(3.51) $\quad \kappa = \kappa(\Psi_x).$

On the other hand, it is well-known that the covariance Ψ_x is given as a function of κ for a given system,

(3.52) $\quad \Psi_x = \Psi_x(\kappa).$

The values κ and Ψ_x are determined by solving (3.51) and (3.52) simultaneously via the graphical procedure[107,108]. The fact that κ and Ψ_x are determined by the simultaneous equations corresponds to the situation that the linearization-coefficient B is a function of the covariance matrix P which is determined by a differential equation.

In order to expect the desired relation between the classical statistical and stochastic linearizations, we consider the second-order system,

$$(3.53) \qquad \ddot{x} + c\dot{x} + kx + f(x) = \gamma(t),$$

where γ is a stationary Gaussian random process with the following properties:

(i) mean value: $m_\gamma=0$

(ii) auto-correlation function: $\psi_\gamma(\tau)= \exp(-\beta|\tau|)$

(iii) spectral density: $S_\gamma(\lambda)=\frac{2\alpha\beta}{\lambda^2+\beta^2}$,

where α and β are positive constants and λ is the angular frequency. The block diagram of the system (3.53) is illustrated in Fig.3.2. For the system (3.53), since the random disturbance $\gamma(t)$ is stationary Gaussian, we can replace the nonlinear element $f(x)$ of zero-memory type by an equivalent gain κ. Then the equivalent system with equivalent gain κ is given by the equation,

$$(3.54) \qquad \ddot{x} + c\dot{x} + (k+\kappa)x = \gamma(t).$$

The corresponding equivalent linear system to Fig.3.2 is shown in Fig.3.3. Using the equivalent gain κ, the spectral density $S_x(\lambda)$ of the output $x(t)$ is calculated by

$$(3.55) \qquad S_x(\lambda) = \left|\frac{1}{(j\lambda)^2+c(j\lambda)+(k+\kappa)}\right|^2 S_\gamma(\lambda).$$

Then the variance ψ_x of $x(t)$ is evaluated by using the well-known

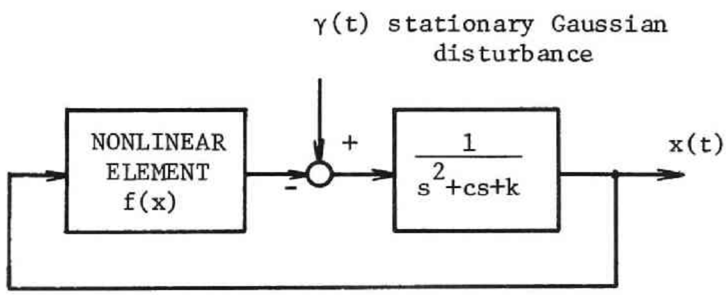

Fig.3.2. Nonlinear system subjected to a stationary Gaussian disturbance.

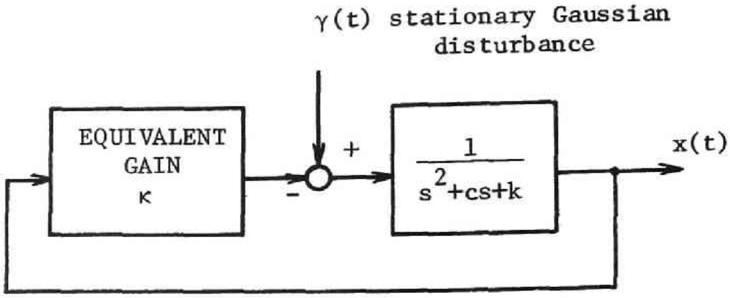

Fig.3.3. Equivalent linear system corresponding to Fig.3.2.

Wiener-Khintchin's formula

$$(3.56)\qquad \psi_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\lambda)\,d\lambda,$$

which yields, after somewhat complicated calculations,

$$(3.57)\qquad \psi_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1}{(j\lambda)^2+c(j\lambda)+(k+\kappa)}\right|^2\frac{2}{\lambda^2+\beta^2}d\lambda$$

$$= \frac{\alpha(\beta+c)}{c(k+\kappa)[(k+\kappa)+\beta(\beta+c)]}.$$

Keeping β/α with a constant, if $\alpha,\beta\to\infty$ in (3.57), then we have

$$(3.58)\qquad \psi_x = \frac{\sigma^2}{2c(k+\kappa)},$$

where $\sigma^2=2\alpha/\beta$* which equal to the variance parameter of the Brownian motion process $w(t)=\int^t\gamma(\tau)d\tau$. Equation (3.58) gives the stationary value of the variance of x(t) when the system is subjected to a stationary white Gaussian disturbance.

Alternatively, the variance of x(t) can be evaluated by the stochastic linearization technique. By letting $x=x_1$ and $\dot{x}=x_2$, Eq.(3.53) is given by

* The variance parameter σ^2 is given by $\sigma^2=\lim_{\alpha,\beta\to\infty} S_\gamma(\lambda)=\frac{2\alpha}{\beta}$.

$$(3.59)\qquad \begin{cases} dx_1 = x_2 dt \\ dx_2 = [-kx_1 - cx_2 - f(x_1)]dt + dw(t), \end{cases}$$

where $w(t)$ process is related to $\gamma(t)$ by the relation $dw(t)=\gamma(t)dt$. Replacing $f(x_1)$ by $[a+b\{x_1-\bar{x}_1\}]$, we have the equivalent system,

$$(3.60)\qquad \begin{cases} dx_1 = x_2 dt \\ dx_2 = [-(k+b)x_1 - cx_2]dt - (a-b\bar{x}_1)dt + dw. \end{cases}$$

Define the covariance p_{ij} by

$$(3.61)\qquad p_{ij} = E\{(x_i-\bar{x}_i)(x_j-\bar{x}_j)\} \quad (i,j=1,2).$$

Then the covariance equations are

$$(3.62)\qquad \begin{cases} \dfrac{dp_{11}}{dt} = 2p_{12} \\ \dfrac{dp_{12}}{dt} = \dfrac{dp_{21}}{dt} = -(k+b)p_{11} - cp_{12} + p_{22} \\ \dfrac{dp_{22}}{dt} = -2(k+b)p_{12} - 2cp_{22} + \sigma^2. \end{cases}$$

If the process x is assumed to be stationary, then $dp_{11}/dt=dp_{12}/dt=dp_{21}/dt =dp_{22}/dt=0$ and

$$(3.63)\qquad \begin{cases} p_{11} = \dfrac{\sigma^2}{2c(k+b)} \\ p_{12} = p_{21} = 0 \\ p_{22} = \dfrac{\sigma^2}{2c}. \end{cases}$$

Therefore the stationary value of variance p_{11} is given, denoting it simly as p, by

$$(3.64)\qquad p = \frac{\sigma^2}{2c(k+b)}.$$

In comparing the stochastic linearization with the classical statistical one, we can observe from (3.64) and (3.58) that two linearization-coefficients b and κ plays the same role with each other. In order to investigate the relation between the two linearizations in more detail, we need a further discussion.

If the nonlinear function $f(x)$ in (3.53) is given, say a saturat: function, as

$$f(x) = \begin{cases} A & \text{for } x>A \\ x & \text{for } |x|\leq A \\ -A & \text{for } x<A, \end{cases} \tag{3.65}$$

then the equivalent gain κ is obtained by the assumption of Gaussian for $p(x)$, i.e.,

$$p(x) = \frac{1}{\sqrt{2\pi\psi_x}} \exp\{-\frac{x^2}{2\psi_x}\}dx \tag{3.66}$$

From (3.48) and (3.66), we have

$$\begin{aligned} \kappa &= \frac{1}{\psi_x\sqrt{2\pi\psi_x}} \int_{-\infty}^{\infty} xf(x)\exp\{-\frac{x^2}{2\psi_x}\}dx \\ &= \frac{1}{\sqrt{2\pi\psi_x}} \int_{-\infty}^{\infty} -f(x)\,[\exp\{-\frac{x^2}{2\psi_x}\}]'dx \\ &= \frac{1}{\sqrt{2\pi\psi_x}} \int_{-\infty}^{\infty} f'(x)\exp\{-\frac{x^2}{2\psi_x}\}dx, \end{aligned} \tag{3.67}$$

where the last equality follows by the integration by parts and "'" denotes the differentiation with respect to x. Substituting (3.65) int((3.67), we have

$$\begin{aligned} \kappa &= \frac{1}{\sqrt{2\pi\psi_x}} \int_{-A}^{A} 1\cdot\exp\{-\frac{x^2}{2\psi_x}\}dx \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{A}{\sqrt{2\psi_x}}} \exp(-\phi^2)\,d\phi \equiv \mathrm{erf}(\frac{A}{\sqrt{2\psi_x}}).* \end{aligned} \tag{3.68}$$

Parameters κ and ψ_x are simultaneously determined by (3.58) and (3.68).

On the other hand, the linearization-coefficients a and b are determined by (see Appendix A, Table A.1)

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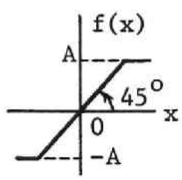
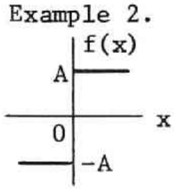
* Error function: $\mathrm{erf}\ x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\lambda^2} d\lambda.$

Table 3.1. Comparison of Stochastic- and Statistical Equivalent-Linearizations

	Statistical Equivalent Linearization	Stochastic Linearization
Linearization	$f(x) = \kappa x(t) + \varepsilon(t)$	$f(x) = a + B\{x-\bar{x}\} + \varepsilon(t)$
Criterion	$E\{\|\varepsilon(t)\|^2\}$	$E\{\|\varepsilon(t)\|^2 \mid x(t_0)=x_0\}$
Coefficient(s)	$\kappa = E\{f(x)x'\}\Psi_x^{-1}$ where Ψ_x=cov.[x]	$a = E\{f(x) \mid x_0\}$ $B = E\{(f-\bar{f})(x-\bar{x})' \mid x_0\}P^{-1}$ where P=cov.$[x \mid x_0]$
pdf (assumed)	$p(x) \sim N[0,\Psi_x]$ $(2\pi)^{-\frac{n}{2}}\lvert\Psi_x\rvert^{-\frac{1}{2}}\exp\{-\frac{1}{2}\|x\|^2_{\Psi^{-1}}\}$	$p\{t,x \mid x_0\} \sim N[\bar{x},P]$ $(2\pi)^{-\frac{n}{2}}\lvert P\rvert^{-\frac{1}{2}}\exp\{-\frac{1}{2}\|x-\bar{x}\|^2_{P^{-1}}\}$
Example 1. 	$\kappa = \mathrm{erf}(\frac{A}{\sqrt{2\psi_x}})$	$a = \frac{1}{2}[(A+\bar{x})\mathrm{erf}(\frac{A+\bar{x}}{2p})-(A-\bar{x})\mathrm{erf}(\frac{A-\bar{x}}{2p})]$ $+\sqrt{\frac{p}{2\pi}}[\exp\{-\frac{(A+\bar{x})^2}{2p}\}-\exp\{-\frac{(A-\bar{x})^2}{2p}\}]$ $b = \frac{1}{2}[\mathrm{erf}(\frac{A+\bar{x}}{\sqrt{2p}})+\mathrm{erf}(\frac{A-\bar{x}}{\sqrt{2p}})]$
Example 2. 	$\kappa = A\sqrt{\frac{2}{\pi\psi_x}}$	$a = A\ \mathrm{erf}(\frac{\bar{x}}{\sqrt{2p}})$ $b = A\sqrt{\frac{2}{\pi p}}\ \exp\{-\frac{\bar{x}^2}{2p}\}$

$$(3.69a)\qquad a = \frac{1}{2}[(A+\bar{x})\mathrm{erf}(\frac{A+\bar{x}}{\sqrt{2p}})-(A-\bar{x})\mathrm{erf}(\frac{A-\bar{x}}{\sqrt{2p}})]$$

$$+\sqrt{\frac{p}{2\pi}}[\exp\{-\frac{(A+\bar{x})^2}{2p}\}-\exp\{-\frac{(A-\bar{x})^2}{2p}\}]$$

$$(3.69b)\qquad b = \frac{1}{2}[\mathrm{erf}(\frac{A+\bar{x}}{\sqrt{2p}})+\mathrm{erf}(\frac{A-\bar{x}}{\sqrt{2p}})].$$

If we assume as a matter of convenience that the mean $\bar{x}$ is identica zero, then we have from (3.69) that

$$(3.70a)\qquad a_0 \triangleq a\big|_{\bar{x}=0} = 0$$

$$(3.70b)\qquad b_0 \triangleq b\big|_{\bar{x}=0} = \mathrm{erf}(\frac{A}{\sqrt{2p}}).$$

Since in this case $p=\psi_x$, this means that the coefficient b_0 is the same as κ and so that the stochastic linearization "degenerates" to the class statistical linearization.

From the above investigation, we conclude that:

(1) If the additive Gaussian disturbance is stationary and if we can assume that the pdf, $p\{t,x|x_0\}$, is Gaussian with zero-mean, then, for the nonlinear element which is of the zero-memory typ and is the odd function, the coefficient a identically equals to zero and b becomes the same form as the statistical-equivale gain κ.

(2) The stochastic linearization technique is an extension of the statistical equivalent linearization technique to the non-stationary Gaussian process and to the nonlinear function whict is not necessarily odd.

(3) The stochastic linearization technique degenerates formally to the statistical equivalent linearization technique if we set $\bar{x}=0$ in the coefficients.

The correspondence of the two linearization techniques are listed in Table 3.1.

CHAPTER 4. SIGNAL DETECTION AND ESTIMATION IN GAUSSIAN NOISE

4.1. Introductory Remarks

Up to the present time, most part of the current researches of filtering theory assumed *a priori* that the waveform of the received signal is perfectly known as a function of time and/or that the signal is generated by a class of dynamical systems whose initial time is preassigned. In practical applications, however, there are many cases where the presence of signal in up-dated observed data may be uncertain or the initial time of the signal may not perfectly be known at the beginning of the estimation process.

The work presented in this chapter is motivated by such applications as the tracking of missiles or airplanes, the orbit determination of spacecrafts, and the estimation of land and/or sea traffic flows. Its objectives are twofold: to solve some specific signal detection problems and to establish a coupled scheme of detection and estimation from the detection-theoretic point of view. The objectives are associated with the problem of extraction of the signal from noise corrupted observed data, where the signal is formed as the output of a stochastic dynamical system whose initial time is unknown.

The signal detection problems are solved in general by computing the well-known likelihood-ratio function in detection theory, accompanied by the state estimation problem[24,154]. In order to solve this estimation problem, it is required to establish an exact mathematical model including its initial time. Even though a mathematical model of dynamical systems is specified by empirical relations, it is almost impossible to compute a likelihood-ratio function unless the initial time of the systems is *a priori* preassigned. It is well-known that the computation of the likelihood-ratio function requires the computation of the state estimation and that these two computations are mutually interrelated. When we compute the state estimation by using filter dynamics, it is indeed a prerequisite to know about the initial time of the dynamical systems. Therefore we need to know the exact initial time of the systems.

However, it goes without saying that errors are inevitable in assigning mathematical models as well as its initial time and that a filter model derived from the inexact dynamical model will degrade the filter performance. In order to see this, let $\tau_0(\omega)$ be an initial time of the dynamical system and take its value at one of possible times, $\{t_0, t_1, \cdots, t_{N-1}\}$. Furthermore, let the symbol H_i be the hypothetical event such that

$$H_i = \{\omega: \tau_0(\omega)=t_i\}, \quad (i=0,1,\cdots,N-1)$$

where ω is the generic point of the probability space Ω. Then the error covariance matrix defined by $Q_i(t|t)\triangleq E\{[x(t)-\hat{x}_i(t|t)][x(t)-\hat{x}_i(t|t)]'|Y_0^t,H_j\}$ is greater than or equal to the covariance matrix $P_j(t|t)\triangleq \mathrm{cov.}[x(t)|Y_0^t,H_j]$ $=E\{[x(t)-\hat{x}_j(t|t)][x(t)-\hat{x}_j(t|t)]'|Y_0^t,H_j\}$, i.e. $Q_i(t|t)\geq P_j(t|t)$, where $\hat{x}_i(t|t)=E\{x(t)|Y_0^t,H_i\}$ is an estimation conditioned by the observed data up to time t, Y_0^t, provided that the initial time is $\tau_0(\omega)=t_i$. This fact means that when the hypothesis H_j is actually true the misled error covariance is always greater than or equal to the covariance based on the true hypothesis. Consequently, in order to perform the detection and estimation procedure, we have to guess the initial time as precise as possible.

Up to the present time, concepts and methods of detection theory have been applied to the signal detection coupled with estimation of signals by many researchers[30,51,52,58-61,85,124,128], forcing us to look deeper into the mathematical aspects of the detection and estimation problems. For example, Lainiotis[85] has established a joint method of Bayesian detection, estimation and identification for nonlinear systems. Jaffer and Gupta[51,52] have developed a Bayes optimum theory of joint detection and estimation of signals in white Gaussian noise by using cost functions that reflect the coupling between the operations of detection and estimation, and established certain explicit relations between the procedures of detection and estimation. Recently, several efforts have been made for the detection problem that are somewhat different from the references [30,51,52,58-61,85,124,128]. Prabhu[165] has proposed a method of detection of a change in system parameters whose probability densities are completely known. In [165], the dynamics is not found which represents possible physical phenomena. Sanyal and Shen[167] and Sanyal[166] have discussed the problem of detection and estimation of an unknown impulse applied at unknown time.

In this chapter, based on the likelihood-ratio concept in the detection theory, a procedure of detection and estimation is proposed which will be shown to be a practical computer implementation for detection strategies, and describe the joint method of detection and estimation.

The problem is briefly stated in Sec.4.2. In Sec.4.3, defining a combined risk, a possible solution is given for a signal detection problem. The solution needs the state estimation procedure. The relations between signal detection and estimation are stated in Sec.4.4. Simulation results are shown in Sec.4.5 to illustrate the proposed method of detection.

4.2. Problem Statement

The observation model is given by

$$(4.1) \qquad dy(t) = \begin{cases} R(t)dv(t) & 0 \le t < \tau_0(\omega) \\ s(t)dt + R(t)dv(t) & \tau_0(\omega) \le t. \end{cases}$$

In (4.1), $s(t)$ is an l-vector signal process; $v(t)$ is a d_1-vector additive noise which is considered to be a Brownian motion process with unit covariance; $y(t)$ is an l-vector observed signal; and $R(t)$ is an $l\times d_1$ known matrix. The time $\tau_0(\omega)$ is the random and unknown time at which the signal $s(t)$ becomes to be observed. The problem is to decide from the observed signal $y(t)$ at which time and what signal is actually transmitted. The model (4.1) is fairly good for a variety of situations of practical applications to the problems of tracking, orbit determination and traffic control, and it also will serve as an archetype for various realistic models. The major oversimplification for many applications is that the time τ_0 and/or signal $s(t)$ are assumed to be known.

The signal process $s(t)$ is given as the output of a dynamic system, i.e.

$$(4.2) \qquad s(t) = H(t)x(t)$$

and

$$(4.3) \qquad dx(t) = A(t)x(t)dt + G(t)dw(t),$$

where $x(t)$ is an n-vector state process ($n\geq l$); $w(t)$ is a d_2-vector Brownian motion process with unit covariance, and is independent of $v(t)$-process; and H, A and G are respectively $l\times n$, $n\times n$ and $n\times d_2$ matrices.

The essential subject of our problem is to construct the method of detection and estimation in order to know whether the signal is really present or not, and to know what is the best estimate of the signal, if it presents. For such a method, it may be required to consider a certain joint detection-estimation procedure[52].

For further development, the following assumptions are made.

(H4.1) For $\tau_0(\omega)\leq t$, equation (4.3) is valid and its solution exists and unique w.p.1.

(H4.2) $\{R(t)R'(t)\}$ is nonsingular.

(H4.3) Given the preassigned interval $[0,T]$, the time $\tau_0(\omega)$ is the random variable such that

$$\tau_0(\omega)\varepsilon I \quad \text{w.p.1},$$

where I is a finite set of the *a priori* known time instants, i.e. $I=\{t_i;\ i=0,1,\cdots,N-1\}$ $(0=t_0<t_1<\cdots<t_{N-1}<t_N=T)$, and

satisfies the conditions of the separability definitions[28].

(H4.4) The *a priori* probabilities are uniform that the signal s(t) to be observed begins with any one of t_i's. In other words, if H_i is the hypothesis that $\tau_0(\omega)=t_i$ $(i=0,1,\cdots,N-1)$, then $P(H_0)=P(H_1)=\cdots=P(H_{N-1})=1/N$.

In the following section, the discussions are forcussed on the detection-estimation method.

4.3. A Multiple Alternative Hypothesis Approach to Signal Detection and Detection Rule

In order to determine if a signal s(t) is present, and if so, to determine which one is the true hypothesis among H_i's $(i=0,1,\cdots,N-1)$, we take an approach of multiple alternative hypothesis test (cf.[154]). At the present time t, based on the observed data $Y_0^t=\{y(s),\ 0\leq s\leq t\}$, the hypotheses are

$$H_{-1}:\quad dy(\tau) = R(\tau)dv(\tau) \qquad 0\leq\tau\leq t$$

$$H_i\ :\quad dy(\tau) = \begin{cases} R(\tau)dv(\tau) & 0\leq\tau<t_i \\ s(\tau)d\tau + R(\tau)dv(\tau), & t_i\leq\tau\leq t \end{cases}$$

where $i=0,1,2,\cdots,k-1$ and $t_{k-1}<t\leq t_k$. The hypothesis H_{-1} is the null hypothesis that $\tau_0(\omega)$ is not in $[0,t$.

The hypothesis test is performed by the following two steps:

Step I. Decide whether the signal is already present or not,

Step II. If the signal is present, accept the likeliest hypothesis among H_i's.

To fix the idea, consider the likelihood-ratio comparing the i-th hypothesis with the null one defined by

$$(4.4)\qquad \Lambda(t,t_i) = \frac{p\{Y_0^t|H_i\}}{p\{Y_0^t|H_{-1}\}},\quad i=0,1,\cdots,k-1$$

where p is the conditional probability density function(pdf). If none of these is greater than a threshold,* we accept H_{-1} (Step I). Otherwise, we accept the hypothesis corresponding to the maximum $\Lambda(t,t_i)$ (Step II). The detection rule can be stated in terms of the Λ as follows: accept H_i if the largest Λ is greater than the threshold and accept H_{-1} otherwise. Thus the hypothesis test terminates at the first time at which the hypothesis H_{-1} is rejected in Step I, and the likeliest hypothesis which corresponds to the maximum likelihood-ratio is accepted. Otherwise, if the hypothesis H_{-1} is accepted in Step I, then the test is continued with the further observation.

According to the Bayes test, consider the following combined risk of detection and estimation for $t_{k-1}<t\leq t_k$:[98,52]

$$(4.5)\qquad \bar{R} = \sum_{i=-1}^{k-1}\sum_{j=-1}^{k-1} \int_{Z_i}\int_{S_{ij}}\int_X D[x(s),\hat{x}_i(s|s),H_j] \times p\{x(s),Y_0,H_j\}dxdsdY_0^t,$$

where $D[\cdot,\cdot,H_j]$ is a scalar-valued cost reflecting the coupling between detection and estimation when actually hypothesis H_j is true; $\hat{x}_i(s|s)$ in the cost D is the optimal estimate of $x(s)$ given that H_i is true, i.e. $\hat{x}_i(s|s)=E\{x(s)|Y_0^s,H_i\}$; $p\{\cdot,\cdot,H_j\}$ is a joint pdf of the state and the observation accompanied with the hypothesis H_j; S_{ij} is the time interval over which D is considered; X is the sample space of x; and Z_i is such the family of Y_0^t in which H_i is accepted that $Z^t=Z_{-1}\oplus Z_0\oplus\cdots\oplus Z_{k-1}$, where Z^t is the observation data space of Y_0^t and $\oplus$ is the direct sum.

Defining

$$(4.6)\qquad f_{ij}(Y_0^t) = \int_{S_{ij}}\int_X D[x(s),\hat{x}_i(s|s),H_j]p\{x(s)|Y_0^t,H_j\}dxds$$
$$= \int_{S_{ij}} E\{D[x(s),\hat{x}_i(s|s),H_j]|Y_0^t,H_j\}ds$$

and using the Bayes rule to (4.5), equation (4.5) becomes

---

* The threshold is given later in this section, depending on the preassigned costs and the *a priori* probabilities of the hypotheses.

$$(4.7)\qquad \bar{R} = \sum_{i=-1}^{k-1} [\int_{Z_i} \sum_{j=-1}^{k-1} P(H_j) f_{ij}(Y_0^t) p\{Y_0^t|H_j\} dY_0^t.$$

We adopt the two-step procedure for the hypothesis test by minimizing the combined risk given by (4.7).

(i) Step I. Rewrite (4.7) as follows.

$$\begin{aligned}(4.8)\qquad \bar{R} &= \int_{Z_{-1}} \sum_{j=-1}^{k-1} P(H_j) f_{-1j}(Y_0^t) p\{Y_0^t|H_j\} dY_0^t \\ &+ \int_{Z_i} \sum_{j=-1}^{k-1} P(H_j) f_{ij}(Y_0^t) p\{Y_0^t|H_j\} dY_0^t \\ &+ \sum_{\substack{l=0\\ l\neq i}}^{k-1} [\int_{Z_l} \sum_{j=-1}^{k-1} P(H_j) f_{lj}(Y_0^t) p\{Y_0^t|H_j\} dY_0^t] \\ &= \int_{Z_{-1}} \sum_{j=-1}^{k-1} P(H_j) \{f_{-1j}(Y_0^t) - f_{ij}(Y_0^t)\} p\{Y_0^t|H_j\} dY_0^t \\ &+ \int_{Z^t} \sum_{j=-1}^{k-1} P(H_j) f_{ij}(Y_0^t) p\{Y_0^t|H_j\} dY_0^t \\ &- \int_{Z_{-(i-1)}} \sum_{j=-1}^{k-1} P(H_j) f_{ij}(Y_0^t) p\{Y_0^t|H_j\} dY_0^t \\ &+ \sum_{\substack{l=0\\ l\neq i}}^{k-1} [\int_{Z_l} \sum_{j=-1}^{k-1} P(H_j) f_{lj}(Y_0^t) p\{Y_0^t|H_j\} dY_0^t],\end{aligned}$$

where $Z_{-(i-1)} = Z^t - (Z_i \oplus Z_{-1})$. In (4.8), if $Z_{-(i-1)}$ is determined to be constant, then the terms except the first term are considered to be constant. Then

$$(4.9)\qquad \bar{R} = \int_{Z_{-1}} \sum_{j=-1}^{k-1} P(H_j) \{f_{-1j}(Y_0^t) - f_{ij}(Y_0^t)\} p\{Y_0^t|H_j\} dY_0^t + \text{const.}$$

By inspection of (4.9) it follows that the detection rule for Step I is stated as

accept H_{-1}, if for all $i=0,1,\cdots,k-1$

$$(4.10)\qquad \sum_{j=-1}^{k-1} P(H_j)\{f_{-1j}(Y_0^t)-f_{ij}(Y_0^t)\}p\{Y_0^t|H_j\} < 0$$

reject H_{-1}, otherwise.

Assume that $\{f_{i\,-1}(Y_0^t)-f_{-1\,-1}(Y_0^t)\}>0$. Then (4.10) is modified as

$$(4.11)\qquad \sum_{j=0}^{k-1} \frac{P(H_j)}{P(H_{-1})} \frac{\{f_{-1j}(Y_0^t)-f_{ij}(Y_0^t)\}}{\{f_{i-1}(Y_0^t)-f_{-1-1}(Y_0^t)\}} \frac{p\{Y_0^t|H_j\}}{p\{Y_0^t|H_{-1}\}} < 1,$$

where the addend in (4.11) is a kind of cost likelihood-ratio[154].

Since $H_{-1}=\bigcup_{\nu=k}^{N-1} H_\nu$, it follows by the assumption (H4.4) that

$$(4.12)\qquad \frac{P(H_{-1})}{P(H_j)} = \sum_{\nu=k}^{N-1} \frac{P(H_\nu)}{P(H_j)} = N-k = \rho_k.$$

Noting (4.4), write the term in (4.11) as

$$(4.13)\qquad \lambda(t,t_i) = \sum_{j=0}^{k-1} \frac{\{f_{-1j}(Y_0^t)-f_{ij}(Y_0^t)\}}{\{f_{i-1}(Y_0^t)-f_{-1-1}(Y_0^t)\}} \Lambda(t,t_i).$$

Combining (4.12) and (4.13) with (4.11), the condition (4.11) is expres

as

$$(4.14)\qquad \lambda(t,t_i) < \rho_k.$$

(ii) Step II. Write (4.7) as

$$(4.15)\qquad \bar{R} = \int_{Z^t} \sum_{j=-1}^{k-1} P(H_j)f_{\nu j}(Y_0^t)p\{Y_0^t|H_j\}dY_0^t +$$

$$+\sum_{\substack{i=-1\\ i\neq\nu}}^{k-1}\left[\int_{Z_i}\sum_{j=-1}^{k-1}P(H_j)\{f_{ij}(Y_0^t)-f_{\nu j}(Y_0^t)\}p\{Y_0^t|H_j\}dY_0^t\right]$$

Then, since the first term is independent of Z_i, $\bar{R}$ is a minimum when Z_i $(i=-1,0,\cdots,k-1;\ i\neq\nu)$ is chosen as the integrand of the second integral is negative for all ν. This corresponds to choosing the hypothesis H_i whenever, for all ν,

$$(4.16)\qquad \sum_{j=-1}^{k-1}P(H_j)\{f_{ij}(Y_0^t)-f_{\nu j}(Y_0^t)\}p\{Y_0^t|H_j\}\leq 0.$$

Rearranging terms in (4.16), we have (see Appendix B)

$$(4.17)\qquad \Pi(t,t_i)\geq\Pi(t,t_\nu),$$

where

$$(4.18)\qquad \Pi(t,t_i)\triangleq[\lambda(t,t_i)-\rho_k]\{f_{i\,-1}(Y_0^t)-f_{-1\,-1}(Y_0^t)\}$$

and $\Pi(t,t_\nu)$ is defined as a similar relation to (4.18).

Then we have:

accept the hypothesis H_i which gives $\max_{t_i}\Pi(t,t_i)$ $(i=0,1,\cdots,k-1)$, and decide that the initial value exists in the interval $[0,t)$ and that $\tau_0(\omega)=t_i$ where t_i corresponds to the maximum Π.

Combining the two steps, the detection rule is stated as follows:

Detection Rule. At the present time t $(t_{k-1}<t\leq t_k)$, according to the following two steps the hypothesis test is performed.

Step I. Accept H_{-1}, if

$$(4.19a)\qquad \max_{t_i}\lambda(t,t_i)<\rho_k\quad(i=0,1,\cdots,k-1)$$

or alternatively

$$(4.19b)\qquad \max_{t_i}\Pi(t,t_i)<0.$$

If H_{-1} is rejected in Step I. Then

Step II. Accept H_i which gives $\max_{t_i}\Pi(t,t_i)$.

If the cost function $f_{ij}(Y_0^t)$ is preassigned as

$$f_{-1i}(Y_0^t)-f_{ii}(Y_0^t) = f_{i\,-1}(Y_0^t)-f_{-1\,-1}(Y_0^t) \quad (i=0,1,\cdots,k-1) \tag{4.20a}$$

and

$$f_{-1j}(Y_0^t) = f_{ij}(Y_0^t) \quad (j\neq i,\ j=0,1,\cdots,k-1), \tag{4.20b}$$

then $\lambda(t,t_i)$ given by (4.13) becomes simply $\Lambda(t,t_i)$, so that the above hypothesis test reduces to the test given by the principle of maximum likelihood.

Detection Rule. (Special Case) If the cost functions $f_{ij}(Y_0^t)$ are preassigned as (4.20a) and (4.20b), then

Step I. Accept H_{-1}, if

$$\max_{t_i} \Lambda(t,t_i) < \rho_k \quad (i=0,1,\cdots,k-1) \tag{4.21}$$

Step II. Accept H_i which gives $\max_{t_i} \Lambda(t,t_i)$.

If once the decision is made that the hypothesis, say H_i, is true, then the other hypotheses H_ν ($\nu=0,1,\cdots,N-1;\ \nu\neq i$) are rejected. This situation implies that the estimation $\hat{x}_i(t|t)$ is true and the other estimations, $\hat{x}_\nu(t|t)$, are rejected by virtue of H_i, and that after the time t_D where the decision was made, $\hat{x}_j(t|t)$ is adopted as the optimal estimation to the control scheme. Therefore the obtained estimation is a kind of detection-directed estimation with estimate rejection in the sense of Middleton and Esposito[98].

With the help of Fig.4.1, the detection procedure is as follows:

(i) Preassign the cost D in (4.5).

(ii) Obtain a newly observed data $dy(t)$, and compute the likelihood-ratio function $\Lambda(t,t_i)$ and $\Pi(t,t_i)$ by (4.22) or (4.23) (to be given below) and by (4.18). Check, in Step I of Detection Rule, whether $\Pi(t,t_i)$ is negative or not. If Π is negative, decide that the signal is not yet present, and repeat the calculations of Λ and Π.

(iii) If otherwise, proceed to Step II and accept the hypothesis H_i that maximizes the corresponding $\Pi(t,t_i)$ with respect to t_i.

(iv) Choose $\hat{x}_i(t|t)$ by virtue of H_i in the step (iii), rejecting

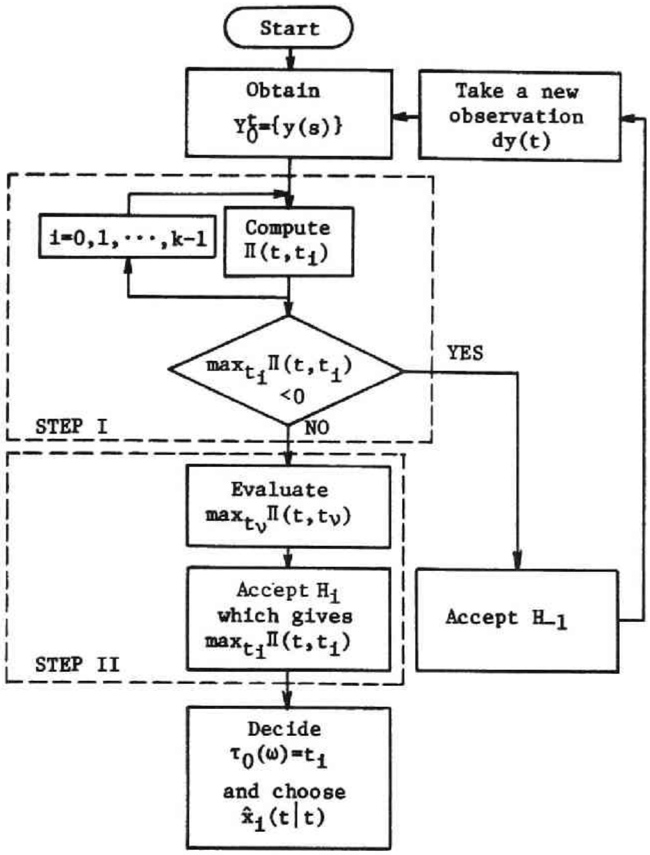


Fig.4.1. Flow diagram for signal detection.

the other estimates $\hat{x}_\nu(t|t)$ $(\nu\neq i)$.

4.4. Relation between Detection and Estimation

For the computation of $\lambda(t,t_i)$ or $\Pi(t,t_i)$ in Detection Rule, it is required to compute the likelihood-ratio $\Lambda(t,t_j)$ defined by (4.4). Starting with the definition (4.4), it is verified that $\Lambda(t,t_j)$ is given by

(4.22) $\qquad \Lambda(t,t_j) = \exp\{\int_{t_j}^{t}\hat{x}_j'(s|s)H'(s)\{R(s)R'(s)\}^{-1}dy(s) -$

$$-\frac{1}{2}\int_{t_j}^{t}\|H(s)\hat{x}_j(s|s)\|^2_{\{R(s)R'(s)\}^{-1}}ds\},$$

$$(t_j\leq t_{k-1}<t\leq t_k;\ j=0,1,\cdots,k-1)$$

where $\hat{x}_j(t_j|t_j)=\hat{x}_0$ (preassigned const.). It also verified that (4.22) is the unique solution of the following stochastic differential equation:

$$(4.23)\qquad d\Lambda(t,t_j) = \Lambda(t,t_j)\hat{x}_j(t|t)H'(t)\{R(t)R'(t)\}^{-1}dy(t)$$

$$\Lambda(t_j,t_j) = 1.$$

The detailed aspect of deriving (4.22) is carried out in Appendix C.

It is noted that in order to calculate $\Lambda(t,t_j)$, $\hat{x}_j(s|s)$ $(t_j\leq s\leq t)$ is required which is the solution of the well-known Kalman-Bucy filter[69],

$$(4.24)\qquad d\hat{x}_j(s|s) = A(s)\hat{x}_j(s|s)ds + P_j(s|s)H'(s)\{R(s)R'(s)\}^{-1} \times\{dy(s)-H(s)\hat{x}_j(s|s)dt\}$$

$$(4.25)\qquad \frac{dP_j(s|s)}{ds} = A(s)P_j(s|s) + P_j(s|s)A'(s) + G(s)G'(s) - P_j(s|s)H'(s)\{R(s)R'(s)\}^{-1}H(s)P_j(s|s),$$

where $P_j(s|s)=\mathrm{cov.}[x(s)|Y_0^s,H_j]$. This situation tells us that the two operations, detection and estimation, are not separated but are "strongly" coupled (cf. Middleton and Esposito[98]; Jaffer and Gupta[51,52]; Lainiotis[85]).

4.5. Simulation Results

In order to examine the proposed method of the detection rule, let us study an example of digital simulations.

System models. Let us consider the one-dimensional case where the observation process is given by

$$(4.26)\qquad dy(t) = \begin{cases} r\,dv(t) & 0\leq t<\tau_0 \\ s(t)dt + r\,dv(t), & \tau_0\leq t \end{cases}$$

and where $s(t)=hx(t)$ and $x(t)$ is generated by

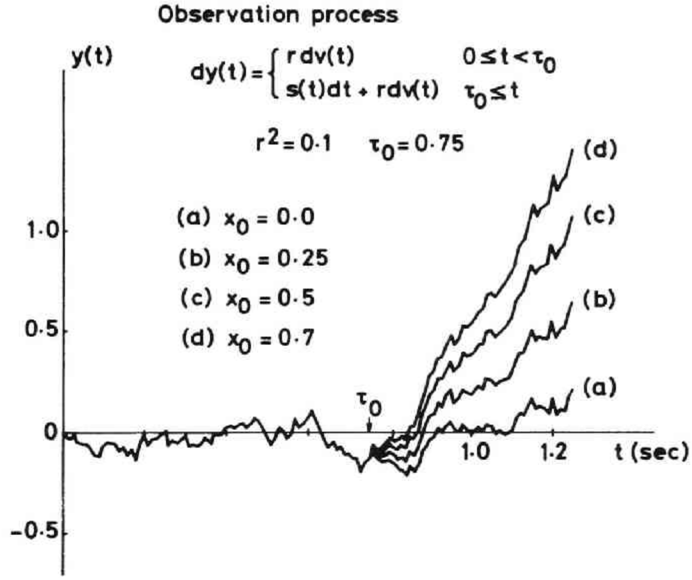


Fig.4.2. Sample processes of observation y(t).

$$(4.27) \qquad dx(t) = ax(t)dt + gdw(t), \quad x(\tau_0) = x_0 \quad (\tau_0 \le t).$$

In digital simulation studies, the true value of τ_0 was set τ_0=0.75(sec), and the time interval in which τ_0 exists was [0,T]=[0,1.25] (sec) which was equi-divided into 25 intervals (N=25) by the times t_i ($i=0,1,\cdots,25$). Each parameter was set as r^2=0.1, g^2=0.2, a=1.0 and h=3.0, and the step-size of time was taken to be dt=0.005(sec). Figure 4.2 shows sample values of the observation process y(t) for the four different initial values: (a) x_0=0.0, (b) x_0=0.25, (c) x_0=0.50 and (d) x_0=0.70.

The estimation $\hat{x}_j(t|t)$ which is necessary to compute the likelihood-ratio Λ is recursively obtained by

$$(4.28) \qquad d\hat{x}_j(t|t) = a\hat{x}_j(t|t)dt + p_j(t|t)hr^{-2}\{dy(t)-h\hat{x}_j(t|t)dt\}$$

$$\hat{x}_j(t_j|t_j) = \hat{x}_0 \qquad (j=0,1,\cdots)$$

$$(4.29) \qquad dp_j(t|t)/dt = 2ap_j(t|t) + g^2 - h^2r^{-2}p_j^2(t|t)$$

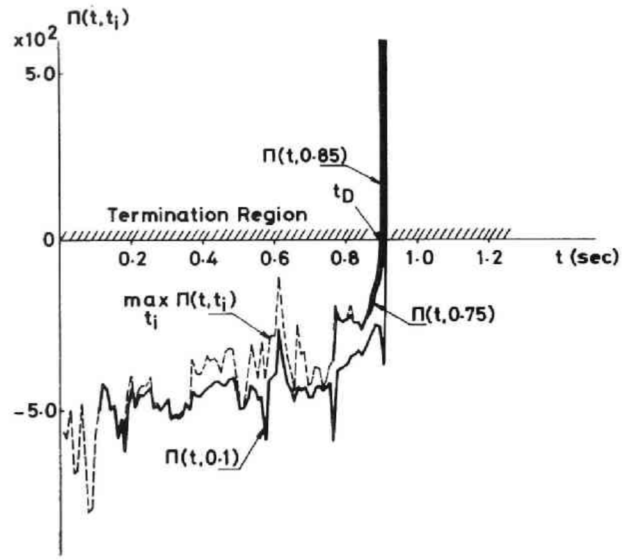


Fig.4.3(a). $\Pi(t,t_i)$-run.

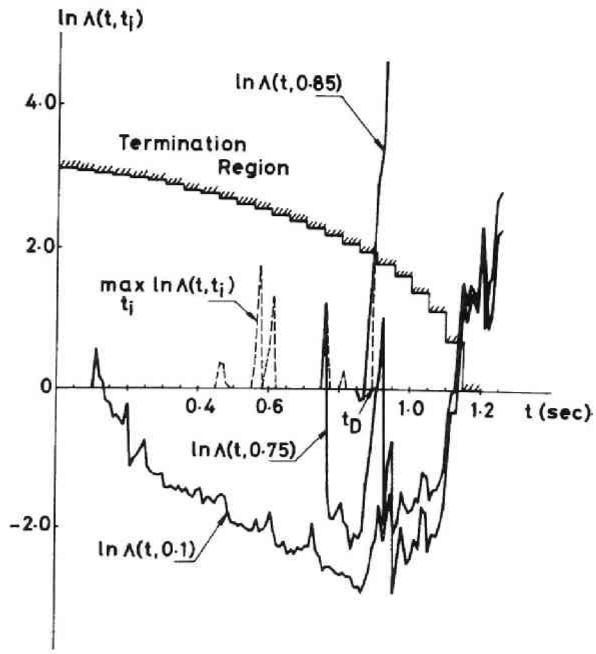


Fig.4.3(b). $\ln \Lambda(t,t_i)$-run.

Fig.4.3. Sample runs of $\Pi(t,t_i)$ and $\ln \Lambda(t,t_i)$ for $(S/N)_I=2.5$.

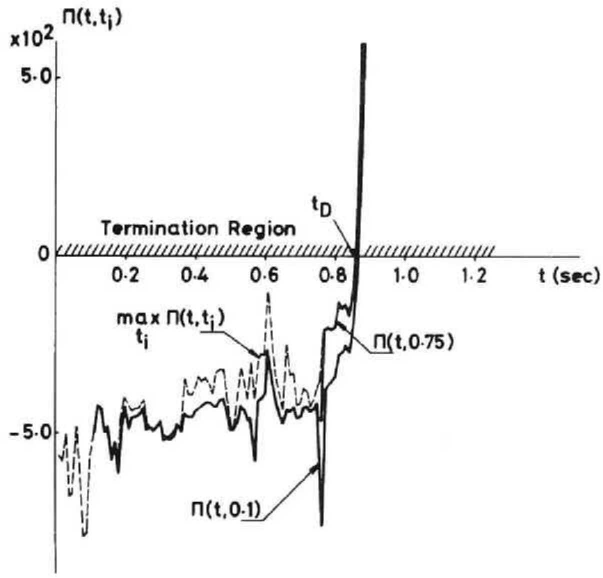


Fig.4.4(a). $\Pi(t,t_i)$-run.

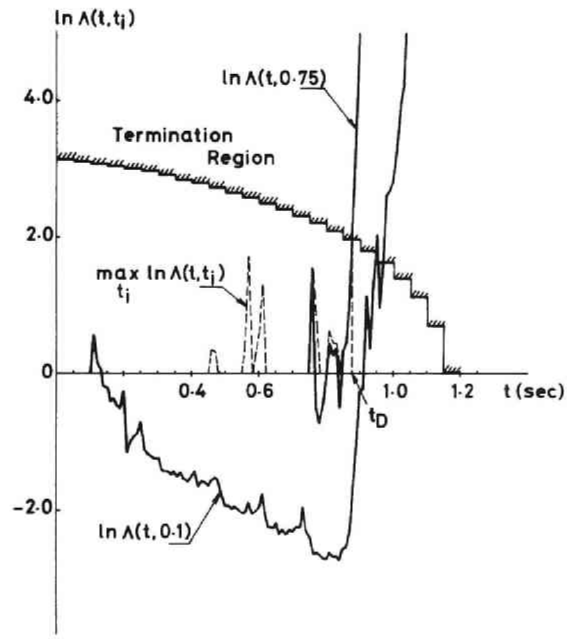


Fig.4.4(b). ln $\Lambda(t,t_i)$-run.

Fig.4.4. Sample runs of $\Pi(t,t_i)$ and ln $\Lambda(t,t_i)$ for $(S/N)_I=7$.

$$p_j(t_j|t_j) = \text{cov.}[x_0|H_j] = p_{j0},$$

where the initial values were given as x_0=1.0 and p_{j0}=1.0 for all j.

Cost assignments. The cost function D and the interval S_{ij} in (4.5) are defined in Appendix D. Hence, $f_{ij}(Y_0^t)$ given by (4.6) were

$$f_{-1\,-1}(Y_0^t) = 0 \qquad f_{-1j}(Y_0^t) = c_1(t-t_j)\{\hat{x}_j^{\,2}(t|t)+p_j(t|t)\}$$

$$f_{i\,-1}(Y_0^t) = \frac{c_2}{2}\,[(T-t)\hat{x}_i^{\,2}(t|t) + (T+t)\{p_0+[\hat{x}_0-\hat{x}_i(t|t)]^2\}]$$

$$f_{ij}(Y_0^t) = c_3T_1q_{ij}(t|t),$$

where

$$q_{ij}(t|t) = p_j(t|t) + [\hat{x}_j(t|t)-\hat{x}_i(t|t)]^2.$$

In the simulation experiments, c_1=60, c_2=c_3=1 and T_1=T=1.25.

Simulation results. Equations (4.26) to (4.29) were simulated on a digital computer. Solving (4.23) for the likelihood-ratio $\Lambda(t,t_i)$, $\Pi(t,t_i)$ which is defined by (4.18) was calculated with use of the costs assigned above. Figures 4.3(a) and 4.4(a) illustrate the results of $\Pi(t,t_i)$ for x_0=0.25 and x_0=0.70. In Fig.4.3(a), only three typical runs are shown for $\Pi(t,0.1)$, $\Pi(t,0.75)$ and $\Pi(t,0.85)$ which correspond to respective hypotheses $H_{0.1}$, $H_{0.75}$ and $H_{0.85}$. In the figure, by tracing the history of $\max_{t_i} \Pi(t,t_i)$ (shown by a dotted line), it is observed that (Step I) it becomes positive at time 0.90(sec), that is, the decision was made at t_D=0.90(sec), and further that (Step II) the hypothesis $H_{0.85}$ can be accepted because $\Pi(t,0.85)$ gives the maximum of Π. As the true value of τ_0 was 0.75, the detection error was 0.10(sec).

Figures 4.3(b) and 4.4(b) shows the runs of log-likelihood ratio $\ln\Lambda(t,t_i)$, corresponding to the parameters x_0 as Figs.4.3(a) and 4.4(a). For these runs, the detection rule for special case was used. In the figures the shaded line shows the threshold $\ln\rho_k$.*

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* The detection rule $\Lambda(t,t_i)\gtrless\rho_k$ is equivalent to $\ln\Lambda(t,t_i)\gtrless\ln\rho_k$.

Table 4.1. Detection Results of Hypothesis Test*

$(S/N)_I$	Detection Rule	Accepted Hypothesis	Decision Time t_D	Detection Error	Delay of Decision
(a) 0	max Π	$H_{0.85}$	0.905	0.100	0.155
	max Λ	$H_{0.85}$	0.905	0.100	0.155
(b) 2.5	max Π	$H_{0.85}$	0.900	0.100	0.150
	max Λ	$H_{0.85}$	0.895	0.100	0.145
(c) 5	max Π	$H_{0.80}$	0.875	0.050	0.125
	max Λ	$H_{0.85}$	0.885	0.100	0.135
(d) 7	max Π	$H_{0.75}$	0.855	no error	0.105
	max Λ	$H_{0.75}$	0.875	no error	0.125

* In the digital simulations, the true hypothesis was $H_{0.75}$.

Comparative aspects are given in Table 4.1 for the two detection rules given in Sec.4.3, with the other simulation results. For convenience of discussions, let us define the following ratio similar to the signal-to-noise ratio by

$$(4.30) \qquad (S/N)_I \triangleq \left| \frac{s(\tau_0)dt}{(rdv)^2} \right| \cong \left| \frac{hx_0}{r^2} \right|.$$

Several facts are pointed out from Table 4.1. First, in order to make the decision sufficient informations are needed regardless of the ratio $(S/N)_I$. Second, the detection error becomes smaller as $(S/N)_I$ becomes large. This means that the larger the ratio becomes, the more detectable does the signal become. Moreover, it is seen that the detection rule for max Π gives better consequences for all ratios than one for max Λ; that is, the detection errors are less smaller than the other. This is due to the fact that the detection rule using Π considers the costs reflecting detection and estimation and the other does not.

From the results obtained it is concluded that the proposed detection rule performs well and is useful for detection of the signal which is generated by a class of dynamical systems with unknown initial time.

4.6. Discussions and Summary

Formulating a multiple alternative hypothesis test, a solution of the method has been presented for signal detection generated by the dynamical system whose initial time is unknown. The estimation of the signal is performed by the detection-theoretic approach; i.e. only the estimation for which the decision is made is accepted and rest are rejected. An example is given of the application of the proposed detection rule to the signal detection, indicating its feasibility to engineering problems.

In this chapter, for the purpose of better understanding of the problem, dynamics of the system and observation are limitted to the linear case. When one or both of the dynamics are nonlinear, then the nonlinear filtering theory is required. The filtering problem of nonlinear systems is the topics in the following chapter.

CHAPTER 5. STATE ESTIMATION FOR NONLINEAR DYNAMICAL SYSTEMS

5.1. Introductory Remarks

When we want to design a control system, the designer has to establish first a procedure to nonlinear filtering as pointed out in Chap.1, Sec.1.2. In Sec.5.2 to Sec.5.4, the author establishes the approximate filter dynamics based on the stochastic linearization technique reviewed in Chap.3 for the nonlinear systems with state-independent and/or state-dependent noise or under state-dependent observation noise whose models are given in Chap.2, Sec.2.3.[126,129-133,135,136,140,143] Some comparative discussions of the approximate filter dynamics obtained here with another approximate filter dynamics based on the Taylor series expansion[111] are demonstrated, including numerical aspects performed by digital simulation studies. Futhermore, in Sec.5.5, an analytical study for performance evaluation is developed in order to provide deeper insight into the ramifications of approximation techniques with a variety of digital simulations[134], and the proposed method of state estimation is particularly emphasized.

5.2. State Estimation for Nonlinear Systems with State-Independent Noise

In this section, an approximate filter dynamics is given for nonlinear systems with state-independent noise. The mathematical model is specified by the system $_{1F}$ defined in Def.2.2 (Chap.2, Sec.2.3), that is, the dynamical system and the observation processes are respectively represented by

$$\left.\begin{array}{ll}(5.1) & dx(t) = f[t,x(t)]dt + G(t)dw(t), \quad x(t_0) = x_0 \\ (5.2) & dy(t) = h[t,x(t)]dt + R(t)dv(t), \quad y(t_0) = 0.\end{array}\right\}: \Sigma_{1F}$$

Expanding the nonlinear function f in (5.1) and using the stochastic linearization reviewed in Sec.3.2, Chap.3, we have

$$f[t,x(t)] = a(t) + B(t)\{x(t)-\hat{x}(t|t)\} + e(t), \tag{5.3}$$

where $e(t)$ denotes the collection of n-dimensional vector terms. In (5.3), $a(t)$ and $B(t)$ are coefficients of the expansion determined by the specific way that the conditional expectation of the squared norm of $e(t)$ conditioned by $\mathcal{Y}_t$, $\mathcal{E}\{\|e(t)\|^2|\mathcal{Y}_t\}$, becomes minimal with respect to $a(t)$ and $B(t)$. The necessary and sufficient conditions for $\min_{a(t),B(t)}\mathcal{E}\{\|e(t)\|^2|\mathcal{Y}_t\}$ are given by

$$a(t) = \mathcal{E}\{f[t,x(t)]|\mathcal{Y}_t\} \triangleq \hat{f}[t,x(t)] \tag{5.4a}$$

$$B(t) = \mathcal{E}\{[f[t,x(t)]-\hat{f}[t,x(t)]][x(t)-\hat{x}(t|t)]'|\mathcal{Y}_t\}P^{-1}(t|t), \tag{5.4b}$$

where

$$P(t|t) = \text{cov.}[x(t)|\mathcal{Y}_t]. \tag{5.4c}$$

Using $a(t)$ and $B(t)$ determined by (5.4), (5.1) can be approximated by the following quasi-linear stochastic differentials of Itô-type:

$$dx(t) = B(t)x(t)dt + \{a(t)-B(t)\hat{x}(t|t)\}dt + G(t)dw(t). \tag{5.5}$$

The same procedure of the linearization is applicable to the observation process given by (5.2). Through the expansion of the function h in the form,

$$h[t,x(t)] = h_1(t) + H_2(t)\{x(t)-\hat{x}(t|t)\} + e_h(t), \tag{5.6}$$

the following conditions can easily be obtained so as to minimize

$E\{\|e_h(t)\|^2|Y_t\}$ with respect to $h_1(t)$ and $H_2(t)$:

(5.7a) $$h_1(t) = E\{h[t,x(t)]|Y_t\} \triangleq \hat{h}[t,x(t)]$$

and

(5.7b) $$H_2(t) = E\{[h[t,x(t)]-\hat{h}[t,x(t)]][x(t)-\hat{x}(t|t)]'|Y_t\}P^{-1}(t|t).$$

For the observation process (5.2), we have

(5.8) $$dy(t) = H_2(t)x(t)dt + \{h_1(t)-H_2(t)\hat{x}(t|t)\}dt + R(t)dv(t).$$

We assume that the conditional pdf $p\{x(t)|Y_t\}$ is Gaussian with the mean value $\hat{x}(t|t)$ and the covariance matrix $P(t|t)$, i.e.

(5.9) $$p\{x(t)|Y_t\} = (2\pi)^{-\frac{n}{2}}|P(t|t)|^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}\|x(t)-\hat{x}(t|t)\|^2_{P^{-1}(t|t)}\}.$$

With the help of (5.9), both $a(t)$ and $B(t)$ can be obtained in the form,

(5.10) $$a(t) = a(t,\hat{x}(t|t),P(t|t))$$

and

(5.11a) $$B(t) = B(t,\hat{x}(t|t),P(t|t))$$

or

(5.11b) $$b_{ij}(t) = \frac{\partial a_i(t)}{\partial \hat{x}_j(t|t)}.$$

Similarly, (5.7a) and (5.7b) become

(5.12a) $$h_1(t) = h_1(t,\hat{x}(t|t),P(t|t))$$

and

(5.12b) $$H_2(t) = H_2(t,\hat{x}(t|t),P(t|t) .$$

A striking fact is that the random variables $a(t)$ and $B(t)$ are not independent but depend mutually on the state estimate $\hat{x}(t|t)$ and the error covariance matrix $P(t|t)$. From this point of view, in reality, more precise symbols, $a(t,\hat{x}(t|t),P(t|t))$ and $B(t,\hat{x}(t|t),P(t|t))$ should be introduced. However, for economy of description, we merely denote these by $a(t)$ and $B(t)$ without indicating the dependence on both $\hat{x}(t|t)$ and $P(t|t)$. Both $h_1(t)$ and $H_2(t)$ also follow this symbolic convention.

The problem considered in this section is to find the minimal variance estimate of the state variable x(t), provided that the process y(s) for $t_0 \leq s \leq t$ is acquired as the observation process. This has already been solved in Ref.[126]. The result is

$$(5.13a)\qquad d\hat{x}(t|t) = \hat{f}[t,x(t)]dt + P(t|t)H_2'(t)\{R(t)R'(t)\}^{-1} \times \{dy(t)-\hat{h}[t,x(t)]dt\}$$

with

$$(5.13b)\qquad \hat{x}(t_0|t_0) = E\{x(t_0)\},$$

where

$$(5.14)\qquad P(t|t) = \text{cov.}[x(t)|\mathcal{Y}_t].$$

This is the solution to the differential equation,

$$(5.15a)\qquad \frac{dP(t|t)}{dt} = B(t)P(t|t) + P(t|t)B'(t) + G(t)G'(t) - P(t|t)H_2'(t)\{R(t)R'(t)\}^{-1}H_2(t)P(t|t)$$

with

$$(5.15b)\qquad P(t_0|t_0) = \text{cov.}[x(t_0)].$$

Equations (5.13) and (5.15) describe the dynamic structure of a quasi-linear estimator for generating a current estimate $\hat{x}(t|t)$ with the preassigned initial values, $\hat{x}(t_0|t_0)$ and $P(t_0|t_0)$.

5.3. State Estimation for Nonlinear Systems with State-Dependent Noise

In this section, an approximate filter dynamics is established for the system whose intensity of the stochastic disturbance depends on the system states. Such systems stated above are called systems "with state-dependent noise." Physical examples of state-dependent noise are found in [94].

For stochastic systems with state-dependent noise, McLane[92] solved a filtering problem of linear dynamical systems with state-dependent noise in a framework of linear filtering theory.

The general structure of the system is the system Σ_{2F} defined in

Def.2.3:

$$\left.\begin{aligned} &(5.16) \quad dx(t) = f[t,x(t)]dt + G_0(t)dw_1(t) \\ &\qquad\qquad + G[t,x(t)]dw_2(t),\ x(t_0) = x_0 \\ &(5.17) \quad dy(t) = h[t,x(t)]dt + R(t)dv(t),\ y(t_0) = 0. \end{aligned}\right\}: \Sigma_{2F}$$

In particular, the state-dependent noise term considered is given by

$$G[t,x(t)] = \sum_{i=1}^{n} x_i(t)G_i(t), \tag{5.18}$$

where $G_i(t)$ is an $n\times d_2$ parameter matrix. The type given by (5.18) was extensively used in [159], [92] and [93].

Although the system equation (5.16) is the version of the Itô sense, it is well-known that there is another version to (5.16); i.e. if the stochastic equation (2.1) is interpreted in the Stratonovich sense, then the equivalent Itô equation is presented, in a component-wise one, by (see Sec.2.2, Chap.2)

$$\begin{aligned} dx_i(t) = {} & [f_i(t,x) + \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{d_2}[G(t,x)]_{kj}\frac{\partial}{\partial x_k}[G(t,x)]_{ij}]dt \\ & + \sum_{j=1}^{d_1}[G_0(t)]_{ij}dw_{1j}(t) + \sum_{j=1}^{d_2}[G(t,x)]_{ij}dw_{2j}(t). \end{aligned} \tag{5.19}$$

Excellent discussions of the relation between Itô and Stratonovich stochastic integrals are found in [54, Chap.4]. It is obvious that the difference between (5.16) and (5.19) is the existence of the term in (5.19),

$$\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{d_2}[G(t,x)]_{kj}\frac{\partial}{\partial x_k}[G(t,x)]_{ij}. \tag{5.20a}$$

For convenience of discussion, with the help of (5.18), we shall write (5.20a) as

$$(G^2x)_i \triangleq \sum_{k=1}^{n}\sum_{j=1}^{d_2}\sum_{l=1}^{n}[G_k(t)]_{ij}[G_l(t)]_{kj}x_l(t), \tag{5.20b}$$

and define an n-vector by

$$(5.21) \qquad (G^2x) \triangleq [(G^2x)_1, \cdots, (G^2x)_n]'.$$

Using the relation,

$$(5.20c) \qquad \sum_{k=1}^{n} \sum_{j=1}^{d_2} \sum_{l=1}^{n} [G_k(t)]_{ij} [G_l(t)]_{kj} x_l(t)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{d_2} \sum_{l=1}^{n} [G_l(t)]_{ik} [G_j(t)]_{lk} x_j(t),$$

write (5.21) as

$$(5.22a) \qquad (G^2x) = \tilde{G}^2 x,$$

where $\tilde{G}^2$ is an n×n-matrix whose (i,j)-th element is given by

$$(5.22b) \qquad [\tilde{G}^2]_{ij} = \sum_{k=1}^{d_2} \sum_{l=1}^{n} [G_l(t)]_{ik} [G_j(t)]_{lk}.$$

Bearing in mind (5.22a), it is convenient to express (5.16) and (5.19) in the following form,

$$(5.23) \qquad dx(t) = [f(t,x) + \tfrac{1}{2}\chi \tilde{G}^2 x]dt + G_0(t)dw_1(t)$$

$$+ \sum_{i=1}^{n} x_i(t) G_i(t) dw_2(t),$$

where is a parameter taking its values 0 or 1 and indicates whether the presented stochastic equation might be interpreted in the sense of Itô or of Stratonovich according to χ=0 or χ=1.

Note that if χ=0, then (5.23) is equal to (5.16), or if χ=1, then (5.23) is equal to (5.19). Equation (5.23) is used for presenting the two different models, (5.16) and (5.19).

The initial condition, $x(t_0)$, for (5.23) is assumed to be a random variable having a zero mean and a covariance matrix $P(t_0|t_0)=E\{x(t_0)x'(t_0)\}$....

Applications of the stochastic linearization technique to the functions f and h in (5.19) and (5.17) yield the quasi-linearized stochastic differentials,

$$(5.24)\qquad dx(t) = [B(t)x(t) + \{a(t)-B(t)\hat{x}(t|t)\} + \frac{1}{2}\chi\tilde{G}^2 x]dt$$

$$+ G_0(t)dw_1(t) + \sum_{i=1}^{n} x_i(t)G_i(t)dw_2(t),$$

$$(5.25)\qquad dy(t) = [h_1(t)+H_2(t)\{x(t)-\hat{x}(t|t)\}]dt + R(t)dv(t).$$

We shall proceed to solve the problem including computation of the state estimate $\hat{x}(t|t)$ and the error covariance $P(t|t)$.

By a simple calculation, the term, $B(t)x(t)+\frac{1}{2}\chi\tilde{G}^2x$, in (5.24) is rewritten as follows,

$$(5.26a)\qquad B(t)x(t) + \frac{1}{2}\chi\tilde{G}^2 x = \tilde{B}_\chi(t)x(t),$$

where $\tilde{B}_\chi(t)$ is an n×n-matrix whose (i,j)-th component is defined by

$$(5.26b)\qquad [\tilde{B}_\chi(t)]_{ij} \triangleq [B(t)]_{ij} + \frac{1}{2}\chi[\tilde{G}^2]_{ij}.$$

Let $\tilde{\Phi}(t,t_0)$ be the formal fundamental matrix associated with the homogeneous differential equation, $dx(t)/dt=\tilde{B}_\chi(t)x(t)$. Although (5.24) involves the state-dependent noise term, it is a simple exercise to show that (5.24) is precisely interpreted by

$$(5.27)\qquad x(t) = \tilde{\Phi}(t,t_0)x(t_0) + \int_{t_0}^{t} \tilde{\Phi}(t,s)\{a(s)-B(s)\hat{x}(s|s)\}ds$$

$$+ \int_{t_0}^{t} \tilde{\Phi}(t,s)G_0(s)dw_1(s) + \int_{t_0}^{t} \sum_{i=1}^{n} \tilde{\Phi}(t,s)x_i(s)G_i(s)dw_2(s).$$

Let the second term on the right-hand side of (5.27) be

$$(5.28)\qquad \zeta(t) = -\int_{t_0}^{t} \tilde{\Phi}(t,s)\{a(s)-B(s)\hat{x}(s|s)\}ds.$$

Introducing a new stochastic process $\xi(t)$ defined by

$$(5.29)\qquad \xi(t) = x(t) + \zeta(t),$$

and combining (5.27) with (5.29), it follows that

$$(5.30)\qquad \xi(t) = \tilde{\Phi}(t,t_0)x(t_0) + \int_{t_0}^{t} \tilde{\Phi}(t,s)G_0(s)dw_1(s) +$$

$$+ \int_{t_0}^{t} \sum_{i=1}^{n} \tilde{\Phi}(t,s)[\xi_i(s)-\zeta_i(s)]G_i(s)\,dw_2(s),$$

where the relation $x(t_0)=\xi(t_0)$ has been used. Then the $\xi(t)$-process is of an Itô-type and the stochastic differential is

$$(5.31) \qquad d\xi(t) = \tilde{B}_X(t)\xi(t)dt + G_0(t)dw_1(t)$$

$$+ \sum_{i=1}^{n} \xi_i(t)G_i(t)dw_2(t) - \sum_{i=1}^{n} \zeta_i(t)G_i(t)dw_2(t).$$

On the other hand, it follows from (5.25) that

$$(5.32) \qquad y(t) = \int_{t_0}^{t} H_2(s)x(s)ds + \int_{t_0}^{t} \{h_1(s)-H_2(s)\hat{x}(s|s)\}ds$$

$$+ \int_{t_0}^{t} R(s)dv(s).$$

Let the second term of the right-hand side of (5.32) be $\zeta_y(t)$ and define $\eta_y(t) \triangleq y(t)-\zeta_y(t)$. Then it follows that

$$(5.33) \qquad d\eta_y(t) = H_2(t)x(t)dt + R(t)dv(t), \quad \eta_y(t_0) = 0.$$

Furthermore defining a new stochastic process by its stochastic differential,

$$(5.34a) \qquad d\eta(t) = d\eta_y(t) + H_2(t)\zeta(t)dt, \quad \eta(t_0) = 0,$$

equation (5.34a) becomes

$$(5.34b) \qquad d\eta(t) = H_2(t)\xi(t)dt + R(t)dv(t),$$

where (5.29) and (5.32) have been used. Since the $\zeta(t)$-process is $\mathcal{Y}_t$-measurable, it follows from (5.29) that

$$(5.35) \qquad \hat{\xi}(t|t) \triangleq E\{\xi(t)|\mathcal{Y}_t\} = \hat{x}(t|t) + \zeta(t).$$

Let $\mathcal{H}_t$ be the σ-algebra of ω sets generated by the random variable $\eta(s)$ for $t_0 \le s \le t$. Since the $\eta(t)$-process is $\mathcal{H}_t$-measurable and the $y(t)$-process $\mathcal{Y}_t$-measurable, it follows that

$$(5.36) \qquad E\{\xi(t)|\mathcal{Y}_t\} = E\{\xi(t)|\mathcal{H}_t\} = \hat{\xi}(t|t).$$

We shall consider that the $\xi(t)$-process is the fictitious state variable determined by (5.31) and that (5.34) denotes the observations which are made on the $\xi(t)$-process. This situation implies that the problem is to find the best estimate, $\hat{\xi}(t|t)$, of $\xi(t)$ based on the σ-algebra $\mathcal{H}_t$.

In order to obtain the current estimate $\hat{\xi}(t|t)$, let the optimal estimate of $\xi(t)$ be generated by

$$(5.37)\qquad d\hat{\xi}(t|t) = F(t)\hat{\xi}(t|t)dt + K(t)d\eta(t), \quad \hat{\xi}(t_0|t_0) = 0.$$

The solution of the above-mentioned class of linear filtering problems is achieved by use of the well-known Wiener-Hopf equation, i.e. [69,17]

$$(5.38)\qquad E\{[\xi(t)-\hat{\xi}(t|t)]d\eta'(s)\} = 0$$

for all $t_0 \leq s < t$. Computing the stochastic differential of $\xi(t)-\hat{\xi}(t|t)$ in (5.38) and using the relation $E\{\cdot\}=E\{E\{\cdot|\mathcal{Y}_t\}\}$, it suffices to show that

$$(5.39)\qquad E\{d\xi(t)d\eta'(s)|\mathcal{Y}_t\} = E\{d\hat{\xi}(t|t)d\eta'(s)|\mathcal{Y}_t\}, \quad t_0 \leq s < t.$$

Using (5.31) and invoking the fact that $w_1(t)$ and $w_2(t)$ are independent of $d\eta(s)$ for $s \in [t_0, t)$, the left-hand side of (5.39) becomes

$$(5.40)\qquad E\{d\xi(t)d\eta'(s)|\mathcal{Y}_t\} = E\{\tilde{B}_\chi(t)\xi(t)d\eta'(s)|\mathcal{Y}_t\}dt.$$

On the other hand, by using (5.34) and (5.37), the right-hand side of (5.39) becomes

$$(5.41)\qquad E\{d\hat{\xi}(t|t)d\eta'(s)|\mathcal{Y}_t\} = E\{F(t)\hat{\xi}(t|t)d\eta'(s)|\mathcal{Y}_t\}dt$$
$$+ E\{K(t)H_2(t)\xi(t)d\eta'(s)|\mathcal{Y}_t\}dt,$$

because $v(t)$ and $d\eta(s)$ are mutually independent for $s \in [t_0, t]$. Since $\tilde{B}_\chi(t)$, $H_2(t)$, $F(t)$ and $K(t)$ are $\mathcal{Y}_t$-measurable, it follows from (5.40) and (5.41) that

$$(5.42)\qquad [\tilde{B}_\chi(t)-F(t)-K(t)H_2(t)]E\{\hat{\xi}(t|t)d\eta'(s)|\mathcal{Y}_t\} = 0.$$

Consider the integral form of (5.37),

$$(5.43)\qquad \hat{\xi}(t|t) = \int_{t_0}^{t} A(t,s)d\eta(s),$$

where $A(t,s)$ is an $n \times n$-matrix associated with $F(t)$ and $K(t)$. Combining (5.42) with (5.43), we have

(5.44) $$\int_{t_0}^{t} [\tilde{B}_\chi(t)-F(t)-K(t)H_2(t)]A(t,\tau)E\{d\eta(\tau)d\eta'(s)|\mathcal{Y}_t\} = 0,$$

or equivalently,

(5.45) $$[\tilde{B}_\chi(t)-F(t)-K(t)H_2(t)]A(t,\tau) = 0$$

for $t_0 \leq \tau < t$. Thus

(5.47) $$d\hat{\xi}(t|t) = \tilde{B}_\chi(t)\hat{\xi}(t|t)dt + K(t)\{d\eta(t)-H_2(t)\hat{\xi}(t|t)dt\}.$$

It is a simple exercise to show that the optimal filter gain is given by [69]

(5.48) $$K(t) = P_\xi(t|t)H_2'(t)\{R(t)R'(t)\}^{-1},$$

where $P_\xi(t|t)=\text{cov.}[\xi(t)|\mathcal{Y}_t]$.

Bearing in mind (5.35) and the definition of $P_\xi(t|t)$, it follows that

(5.49) $$P(t|t) \triangleq \text{cov.}[x(t)|\mathcal{Y}_t] = P_\xi(t|t).$$

Substituting (5.34) and (5.48) into (5.47) and using (5.26), (5.28), (5.29), (5.35) and (5.49), we have

(5.50a) $$d\hat{x}(t|t) = [\hat{f}[t,x(t)] + \frac{1}{2}\chi G^2\hat{x}]dt + P(t|t)H_2'(t)\{R(t)R'(t)\}^{-1}\{dy(t)-\hat{h}[t,x(t)]dt\},$$

(5.50b) $$\hat{x}(t_0|t_0) = E\{x(t_0)\} = 0,$$

where the relations (5.4a) and (5.7a) have also been used. The associated error covariance $P(t|t)$ is determined by

(5.51a) $$\frac{dP(t|t)}{dt} = \tilde{B}_\chi(t)P(t|t) + P(t|t)\tilde{B}_\chi'(t) + G_0(t)G_0'(t) + G[Q] - P(t|t)H_2'(t)\{R(t)R'(t)\}^{-1}H_2(t)P(t|t),$$

(5.51b) $$P(t_0|t_0) = \text{cov.}[x(t_0)],$$

where $G[Q]$ is an $n\times n$-matrix defined by

(5.52) $$G[Q] \triangleq E\{(\sum_{i=1}^{n} x_i(t)G_i(t))(\sum_{j=1}^{n} x_j(t)G_j'(t))|\mathcal{Y}_t\} = \sum_{j=1}^{n}\sum_{i=1}^{n} [Q(t|t)]_{ij}G_i(t)G_j'(t).$$

The n×n-matrix Q in (5.51) is defined by

$$Q(t|t) = E\{x(t)x'(t)|Y_t\} \tag{5.53}$$

and determined by

$$\frac{dQ}{dt} = \tilde{B}_X Q + Q\tilde{B}_X' + a\hat{x}' + \hat{x}a' - B\hat{x}\hat{x}' - \hat{x}\hat{x}'B' + G_0G_0' + G[Q], \tag{5.54a}$$

$$Q(t_0|t_0) = E\{x(t_0)x'(t_0)\}. \tag{5.54b}$$

Equations (5.50), (5.51) and (5.54) describe the dynamic structure of an approximate filter for generating a current estimate $\hat{x}(t|t)$ with the given initial values, $\hat{x}(t_0|t_0)$, $P(t_0|t_0)$ and $Q(t_0|t_0)$. From the results obtained it is learned that if the system dynamics is linear and the observation mechanism is also linear, then the filter dynamics coincide with ones obtained in [92] where the stochastic integral is interpreted in the Stratonovich sense.

5.4. State Estimation for Nonlinear Systems under State-Dependent Observation Noise

5.4.1. System Models and Filter Dynamics

This section is concerned with an approximate filter dynamics for nonlinear stochastic systems under noisy observations, where the intensities of the system and observation noises depend on the system state. Physical examples of state-dependent noise can be found in the operation of radar sevo systems, aerospace systems and chemical process control systems (for more details examples, see [94]).

We consider first the system dynamics of the type Σ_{3F} in Def.2.4, that is the system dynamics is a nonlinear vector stochastic differential equation of the form,

$$dx(t) = f[t,x(t)]dt + G_0(t)dw_1(t) + dW_2(t)x(t), \quad x(t_0) = x_0. \tag{5.55}$$

The observations are made at the output of the nonlinear system with additive Gaussian disturbances. The mathematical model is given by

$$(5.56)\qquad dy(t) = h[t,x(t)]dt + R_0(t)dv_1(t) + dV_2(t)r[x(t)],$$
$$y(t_0) = 0,$$

where $r(x)$ is an n-vector valued nonlinear function of x.

The essential aspect of the problem considered here is the existence of the third term of the right-hand side of (5.56) which is regarded as a term of the "state-dependent observation noise." If the state-dependent noise is not involved in an observation channel the nonlinear filtering problem has already been solved in Refs.[115], [78] and [165] and several methods of establishing approximate filter dynamics have been proposed [4,80,111,126]. However, the existence of the state-dependent noise term brings a difficulty to compute the time evolution of conditional pdf of the system state, based on the Bayesian approach. Furthermore, undoubtedly, the resulting filter dynamics does not suffice to realize only the first two moments even if the functions $f(t,x)$ and $h(t,x)$ are linear.

Up to the present time, a few papers deal with the filtering problem of linear dynamical systems[92]. In [92], McLane considered the filtering problem of linear dynamical models with both the state-dependent system and observation noises and reduced the problem to solve the Wiener-Hopf equation under the assumption that the intensity of the state-dependent noise is not so large as the process becomes non-Gaussian.

In the sequel, for convenience of theoretical development, the case where the influence of the state-dependent observation noise is proportional to the system state. In (5.56), this situation implies

$$(5.57)\qquad r(x) = x.$$

Thus, instead of (5.56), the following model is given:

$$(5.58)\qquad dy(t) = h[t,x(t)]dt + R_0(t)dv_1(t) + dV_2(t)x(t).$$

The problem is to find the minimal variance estimate of the state $x(t)$, provided that the process $y(s)$ for $t_0 \le s \le t$ is acquired as the observation process.

With the applications of the stochastic linearization to the functions f and h, the original processes (5.55) and (5.58) are approximated by

$$(5.59) \qquad dx(t) \cong B(t)x(t)dt + \{a(t)-B(t)\hat{x}(t|t)\}dt + G_0(t)dw_1(t) + dW_2(t)x(t),$$

$$(5.60) \qquad dy(t) \cong H_2(t)x(t)dt + \{h_1(t)-H_2(t)\hat{x}(t|t)\}dt + R_0(t)dv_1(t) + dV_2(t)x(t).$$

Although both the nonlinear functions $f(t,x)$ in (5.55) and $h(t,x)$ in (5.58) are respectively approximated by the linear functions, the state-dependent noise terms $dW_2(t)x(t)$ and $dV_2(t)x(t)$ are still remained and these render the processes non-Gaussian. However, when the intensity of the state-dependent noises is small, the stochastic linearization is plausible and we may still assume that the conditional pdf is approximated to be Gaussian with the mean value $\hat{x}(t|t)$ and the covariance matrix $P(t|t)$ as given by (5.9).

Equations (5.59) and (5.60) are the basic stochastic differentials of Itô-type for the development of the following discussions.

In the case where the state-dependent noise terms $dW_2(t)x(t)$ and $dV_2(t)x(t)$ in (5.55) and (5.58) are identically zero, the suboptimal filtering problem is solved in Sec.5.2, via the stochastic linearization technique, and further the filtering problem of the special case where the state-dependent term is given by $(\Sigma_{i=1}^{n} x_i G_i)dw_2(t)$, instead of $dW_2(t)x(t)$, is solved in Sec.5.3.

Based on these researches and the Gaussian approximation (5.9), we may assume that

$$(5.61) \qquad d\hat{x}(t|t) = \hat{f}[t,x(t)]dt + K(t)\{dy(t)-\hat{h}[t,x(t)]dt\},$$

where the $n\times l$-matrix $K(t)$ is determined so as to minimize the conditional expectation of square-norm of the estimation error, $E\{\|x(t)-\hat{x}(t|t)\|^2|\mathcal{Y}_t\}$. Combining equations (5.59), (5.60), (5.61) with (5.14), it follows that the associated error covariance matrix P is the solution determined by

$$(5.62) \qquad \frac{dP}{dt} = BP + PB' - KH_2P - PH_2'K' + G_0G_0' + \Phi[Q] + K\{R_0R_0'+\Lambda[Q]\}K',$$

where $\Phi[Q]$ and $\Lambda[Q]$ are $n\times n$- and $l\times l$-matrices whose respective (i,j)-

element is given by

$$(5.63) \qquad \{\Phi[Q]\}_{ij} = \begin{cases} \sum_{k=1}^{n} \phi_{ik} q_{kk} & \text{for } i=j \\ 0 & \text{for } i \neq j, \end{cases}$$

$$(5.64) \qquad \{\Lambda[Q]\}_{ij} = \begin{cases} \sum_{k=1}^{n} \lambda_{ij} q_{kk} & \text{for } i=j \\ 0 & \text{for } i \neq j, \end{cases}$$

and where q_{ij} is an (i,j)-element of the matrix

$$(5.65) \qquad Q(t|t) \triangleq E\{x(t)x'(t)|\mathcal{Y}_t\},$$

and this satisfies the differential equation,

$$(5.66a) \qquad \frac{dQ}{dt} = BQ + QB' - B\hat{x}\hat{x}' - \hat{x}\hat{x}'B' + a\hat{x}' + \hat{x}a' + G_0G_0' + \Phi[Q],$$

$$(5.66b) \qquad Q(t_0|t_0) = E\{x(t_0)x'(t_0)\}.$$

Here consider the scalar quantity

$$(5.67) \qquad E\{\|x(t)-\hat{x}(t|t)\|^2|\mathcal{Y}_t\},$$

or equivalently, tr.$\{P(t|t)\}$ as a measure of the filter performance. It is obvious that, in (5.62), if the matrices B, H_2, G_0, R_0, $\Phi[Q]$, $\Lambda[Q]$ and $P(t_0|t_0)$ are assumed to be preassigned, then the value of tr.$\{P(t|t)\}$ will depend upon the choice of the filter gain matrix $K(\tau)$ for $t_0 \leq \tau \leq t$, and that, $\min_{K(\tau)}$ tr.$\{P(t|t)\}$ can be evaluated. Thus, the expected question is that, for $t_0 \leq \tau \leq t$, how the matrix $K(\tau)$ should be chosen so as to minimize the "cost functional" (5.67). Although a trial has been made on a linear time-varying dynamical system[3], use will be made of the dynamic programming method in the sequel.

To do this, at present time t, for every fictitious time τ ($t_0 \leq \tau \leq t$), we shall define a minimum cost functional,

$$(5.68) \qquad V(\tau,P_\tau) \triangleq \min_{K(\tau)} \text{tr.}\{P(t|t)\},$$

where $P_\tau = P(\tau|\tau)$. Applying the principle of optimality to the functional, we have

$$(5.69) \qquad V(\tau,P_\tau) = \min_{K(\tau)} \{V(\tau+d\tau, P_\tau+dP_\tau)\}.$$

Expanding the right-hand side of (5.69) into a Taylor series and neglecting the higher-order term than $O(dt^2)$, it follows that

$$(5.70) \qquad V(\tau,P_\tau) = \min_{K(\tau)} \{V(\tau,P_\tau) + \frac{\partial V}{\partial \tau}d\tau + \text{tr.}\{\frac{\partial V}{\partial P_\tau}dP_\tau\}\},$$

where

$$(5.71) \qquad \text{tr.}\{\frac{\partial V}{\partial P_\tau}dP_\tau\} \triangleq \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial V}{\partial p_{ij}}dp_{ij}.$$

Cancellation of the same term $V(\tau,P_\tau)$ from both sides gives

$$(5.72) \qquad -\frac{\partial V}{\partial \tau} = \min_{K(\tau)} \text{tr.}\{\frac{\partial V}{\partial P_\tau}[B_\tau P_\tau + P_\tau B_\tau' - K_\tau H_{2\tau} P_\tau - P_\tau H_{2\tau}' K_\tau' + G_{0\tau} G_{0\tau}' + \Phi[Q]_\tau + K_\tau R_{0\tau} R_{0\tau}' K_\tau' + K_\tau \Lambda[Q]_\tau K_\tau'] ,$$

where the subscript τ indicates the values at time τ. Therefore, with the concept of a gradient matrix[3], from (5.72), we have

$$(5.73) \qquad K(\tau) = P(\tau|\tau)H_2'(\tau)\{R_0(\tau)R_0'(\tau)+\Lambda[Q]_\tau\}^{-1}, \quad t_0 \le \tau \le t.$$

Then, by letting $\tau=t$, the optimal filter gain in (5.61) becomes

$$(5.74) \qquad K(t) = P(t|t)H_2'(t)\{R_0(t)R_0'(t)+\Lambda[Q]_t\}^{-1}.$$

Therefore, combining (5.61) and (5.62) with (5.74), the optimal filter dynamics and the associated error covariance matrix equation are respectively given by

$$(5.75a) \qquad d\hat{x}(t|t) = \hat{f}[t,x(t)]dt + P(t|t)H_2'(t)\{R_0(t)R_0'(t)+\Lambda[Q]\}^{-1} \times\{dy(t)-\hat{h}[t,x(t)]dt\},$$

$$(5.75b) \qquad \hat{x}(t_0|t_0) = E\{x(t_0)\},$$

$$(5.76a) \qquad \frac{dP(t|t)}{dt} = B(t)P(t|t) + P(t|t)B'(t) + G_0(t)G_0'(t) + \Phi[Q] - P(t|t)H_2'(t)\{R_0(t)R_0'(t)+\Lambda[Q]\}^{-1}H_2(t)P(t|t).$$

$$(5.76b)\qquad P(t_0|t_0) = \text{cov.}[x(t_0)].$$

Equations (5.75), (5.76) and also (5.66) describe the dynamic structure of an approximate filter for generating the current estimate $\hat{x}(t|t)$ and the associated error covariance $P(t|t)$ with the given values, $\hat{x}(t_0|t_0)$, $P(t_0|t_0)$ and $Q(t_0|t_0)$ as initial conditions.

5.4.2. An Illustrative Example and Comparative Discussions

For the purpose of exploring the quantitative aspect, we shall consider the one-dimensional case where the nonlinear dynamical system and observation process are respectively given by the following stochastic differential equations:

$$(5.77)\qquad dx = (\alpha x+\varepsilon\beta x^3)dt + xdw_2,$$

$$(5.78)\qquad dy = xdt + xdv_2,$$

where α and β are constants and ε is a sufficiently small parameter, and where, in this example, the portions of state-independent system and observation noises are assumed to be zero. An application of (5.4a) and (5.4b) to the present case gives us (see also Appendix A, Table A.1)

$$(5.79a)\qquad a(t) = \alpha\hat{x} + \varepsilon\beta\hat{x}(\hat{x}^2+3p),$$

$$(5.79b)\qquad b(t) = \alpha + 3\varepsilon\beta(\hat{x}^2+p).$$

Using (5.79a), (5.79b), (5.75a) and (5.76a) the approximate filter dynamics and the associated covariance are determined by

$$(5.80)\qquad d\hat{x} = \{\alpha\hat{x}+\varepsilon\beta\hat{x}(\hat{x}^2+3p)\}dt + p(\lambda q)^{-1}(dy-\hat{x}dt),$$

and

$$(5.81)\qquad \frac{dp}{dt} = 2\{\alpha+3\varepsilon\beta(\hat{x}^2+p)\}p - p^2(\lambda q)^{-1} + \phi q.$$

Furthermore, from (5.66a), it follows that

$$(5.82)\qquad \frac{dq}{dt} = 2\{\alpha+3\varepsilon\beta(\hat{x}^2+p)\}(q-\hat{x}^2) + 2\{\alpha\hat{x}+\varepsilon\beta\hat{x}(\hat{x}^2+3p)\}\hat{x} + \phi q.$$

Equations (5.77) to (5.82) are simulated on a digital computer with the subroutine for the generation of random disturbances $w_2(t)$ and $v_2(t)$. The computer program for the simulation follows completely the description given later in Sec.6.6, where the simulation method is described in detail associated to an optimal control, with a constant partitioned time δ_j=0.01(sec).

The results of simulation studies are shown in Fig.5.1, 5.2 and 5.3 with a variety of parameters, where both the values of α and β were fixed to be -1.00 and 1.00 respectively. The state-dependent system noise covariance was ϕ^2=0.20 for all the experiments and the observation noise covariance was λ^2=0.10 for the experiment shown in Figs.5.1 and 5.2, and λ^2=1.00 for the experiment shown in Fig.5.3. The initial value of the state are approximately assumed to be Gaussian random variables. The true run of the system state and the quasi-linearized run are shown in Figs.5.1(a), 5.2(a) and 5.3(a). The associated $p(t|t)$- and $q(t|t)$-runs in these three experiments are also shown in Figs.5.1(b), 5.2(b) and 5.3(b).

Figures 5.4, 5.5 and 5.6 show the results of another possible method of approximation based on the Taylor series expansion up to the second order[111,126]. The filter equation and the associated covariances are shown in the figures. From a variety of runs shown in these figures, the accuracy of the filter derived by the stochastic linearization method contends with one of the other filter obtained by the Taylor series expansion method.

From these experiments, it can be obtained that as the intensity of the observation noise becomes large the accuracy of the estimation becomes poor (Figs.5.1, 5.3, 5.4 and 5.6) and that as the quantity of the nonlinearity becomes large the accuracies of the estimation and quasi-linear process become wrong (Figs.5.1, 5.2, 5.4 and 5.5). These experiments reveal that the estimation accuracy depends on both nonlinearities and the intensity of observation noise.

We shall proceed to develop comparative discussions of the evaluation of the filter performance by using the Monte Carlo trials. As a qualitative measure of the performance evaluation, we shall consider

$$c(t) = \frac{1}{N} \sum_{i=1}^{N} \{x^{(i)}(t) - \hat{x}^{(i)}(t|t)\}^2, \tag{5.83}$$

where $x^{(i)}(t)$ and $\hat{x}^{(i)}(t|t)$ denote the i-th true sample run of the system

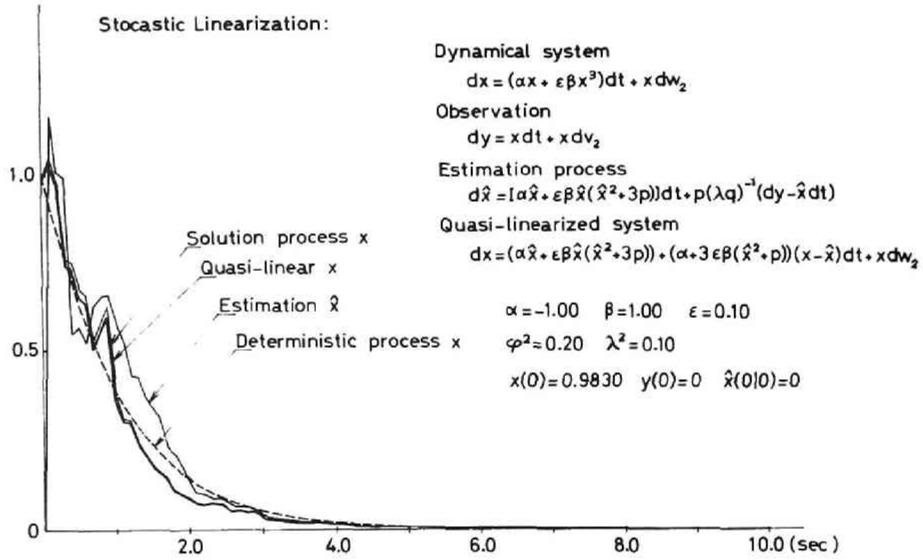


Fig.5.1(a) the output run of the filter

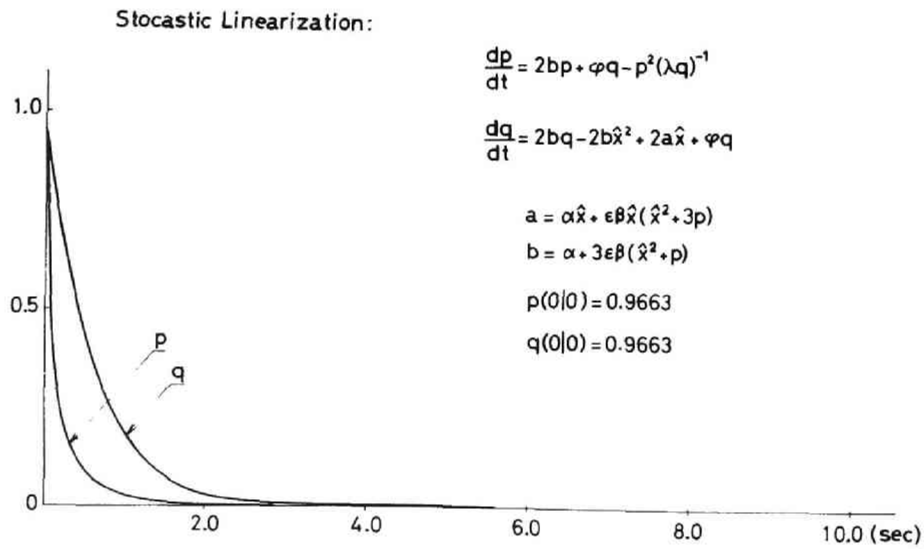


Fig.5.1(b) the output run of the covariance equation

Fig.5.1. A sample path behavior of the approximate filter dynamics (Stochastic linearization) (ε=0.10, ϕ^2=0.20, λ^2=0.10)

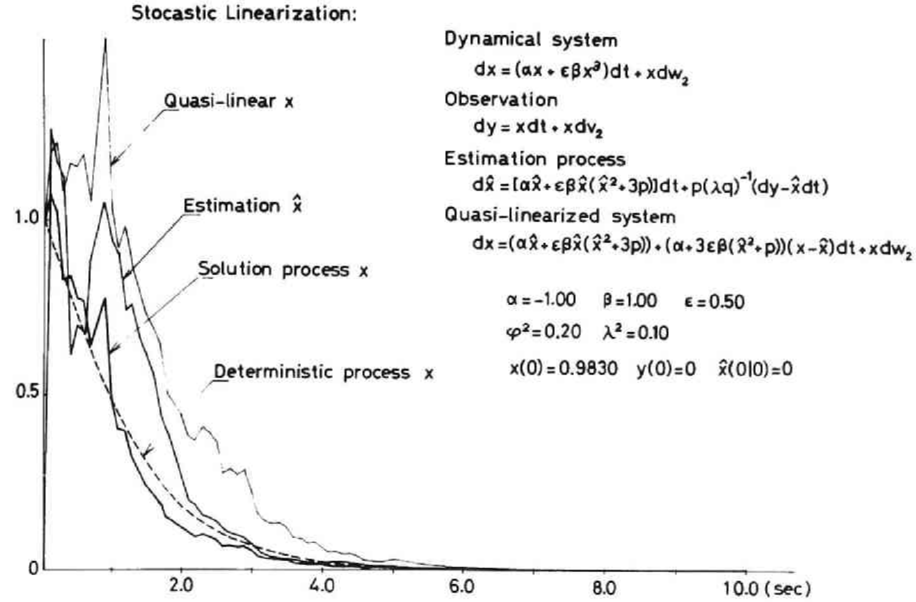


Fig.5.2(a) the output run of the filter

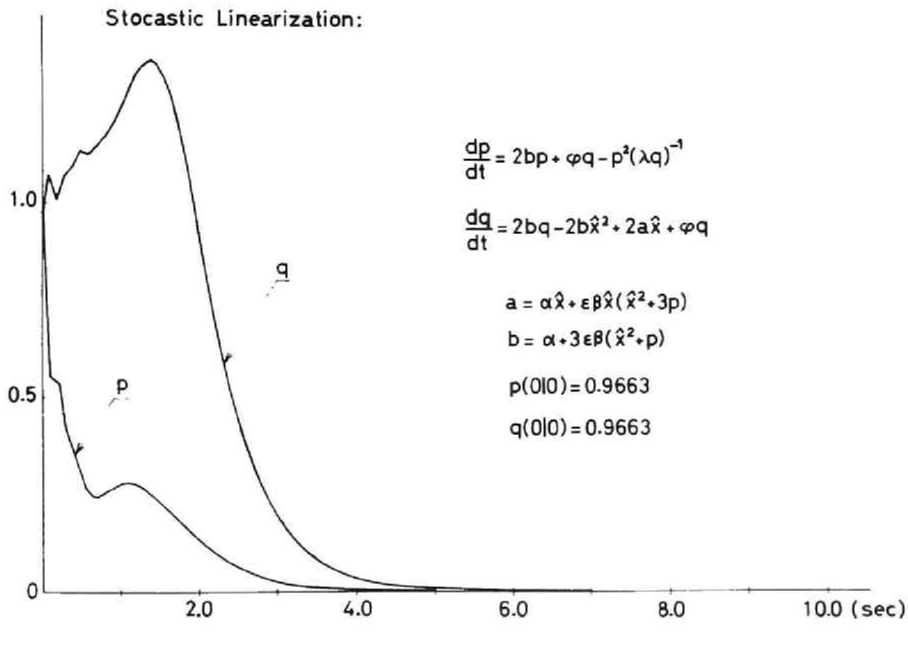


Fig.5.2(b) the output run of the covariance equation

Fig.5.2. A sample path behavior of the approximate filter dynamics (Stochastic linearization) (ε=0.50, ϕ^2=0.20, λ^2=0.10)

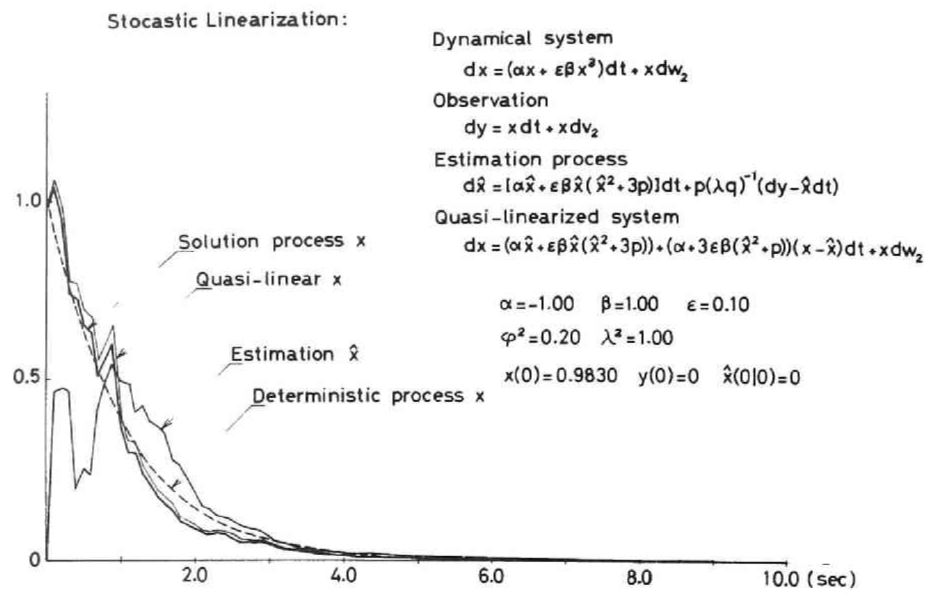


Fig.5.3(a) the output run of the filter

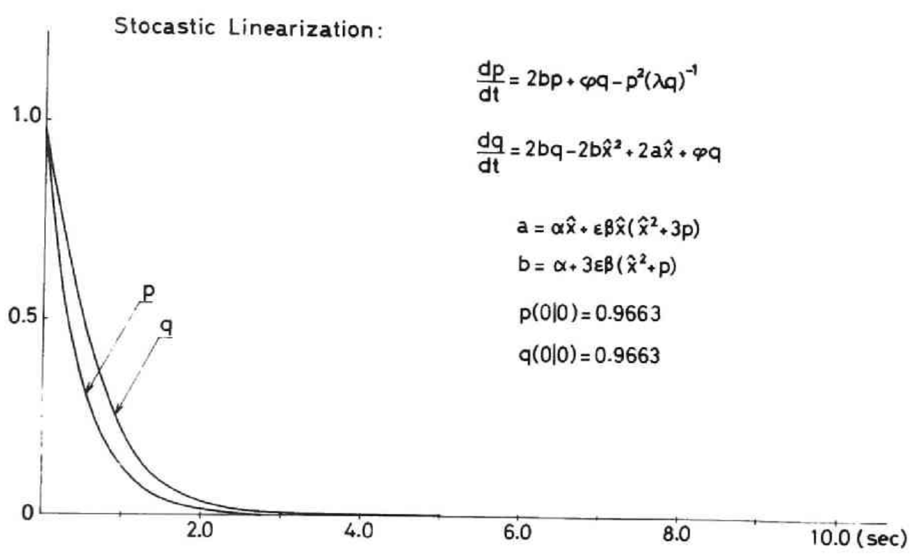


Fig.5.3(b) theoutput run of the covariance equation

Fig.5.3. A sample path behavior of the approximate filetr dynamics (Stochastic linearization) (ε=0.10, ϕ^2=0.20, λ^2=1.00)

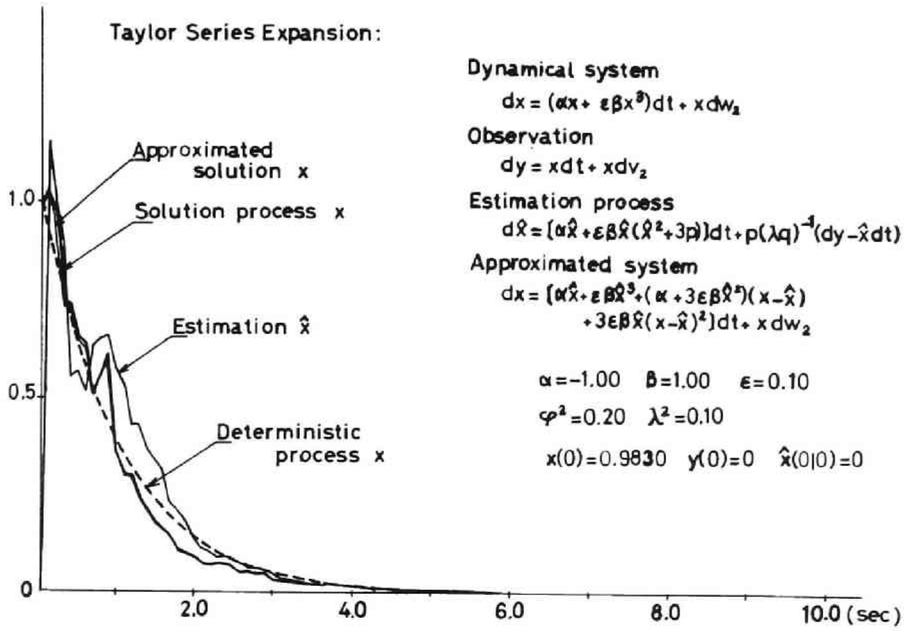


Fig.5.4(a) the output run of the filter

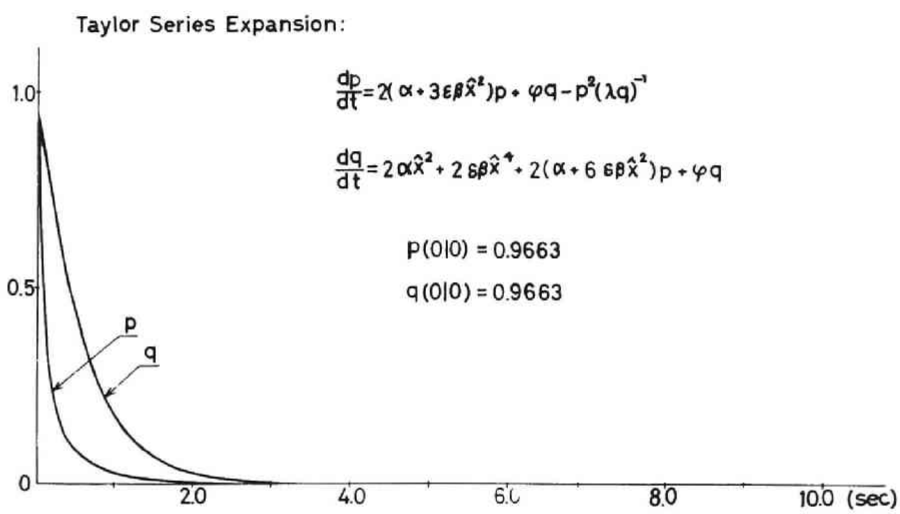


Fig.5.4(b) the output run of the covariance equation

Fig.5.4. A sample path behavior of the approximate filter dynamics (Taylor series expansion) (ε=0.10, ϕ^2=0.20, λ^2=0.10)

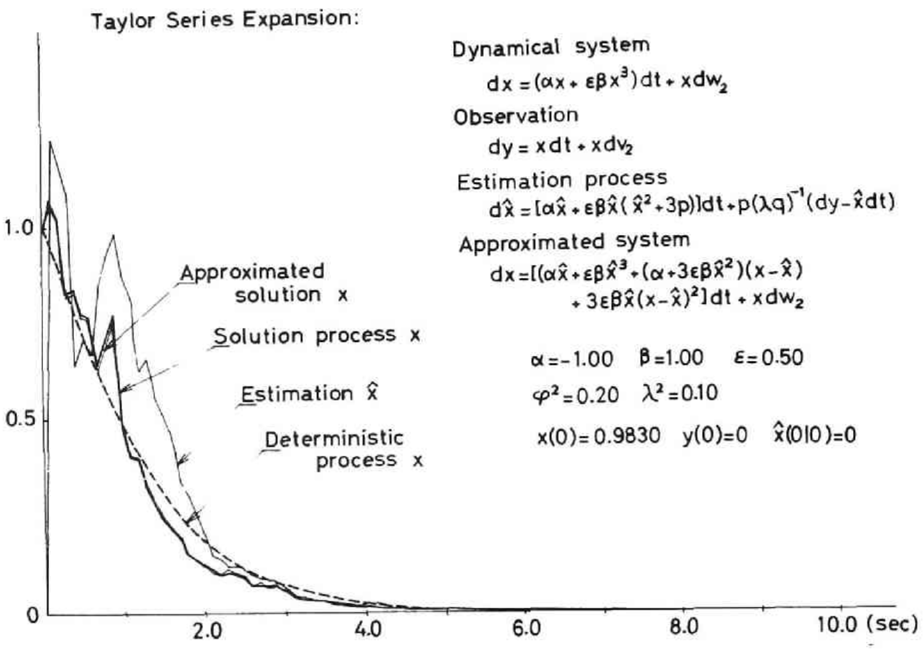


Fig.5.5(a) the output run of the filter

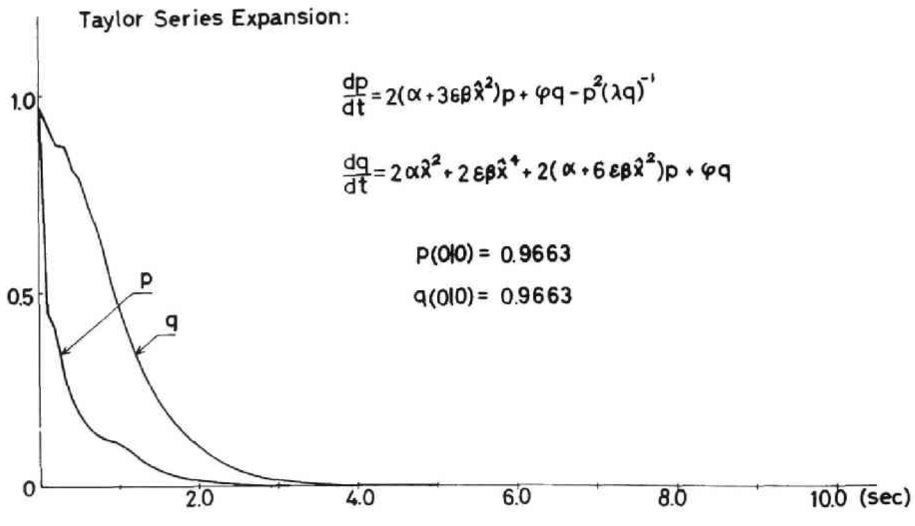


Fig.5.5(b) the output run of the covariance equation

Fig.5.5. A sample path behavior of the approximate filter dynamics (Taylor series expansion) ($\varepsilon=0.50$, $\phi^2=0.20$, $\lambda^2=0.10$)

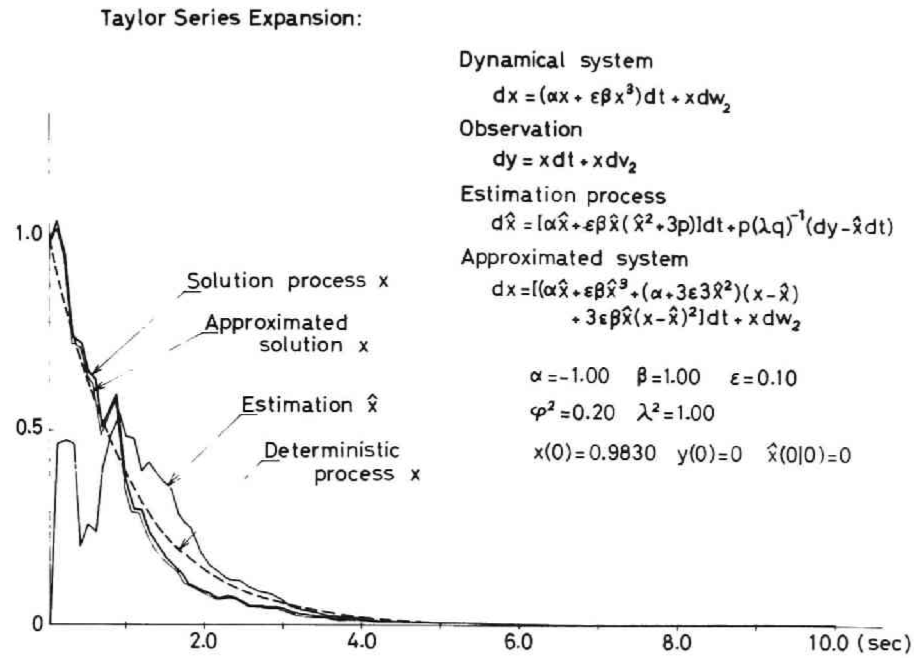


Fig.5.6(a) the output run of the filter

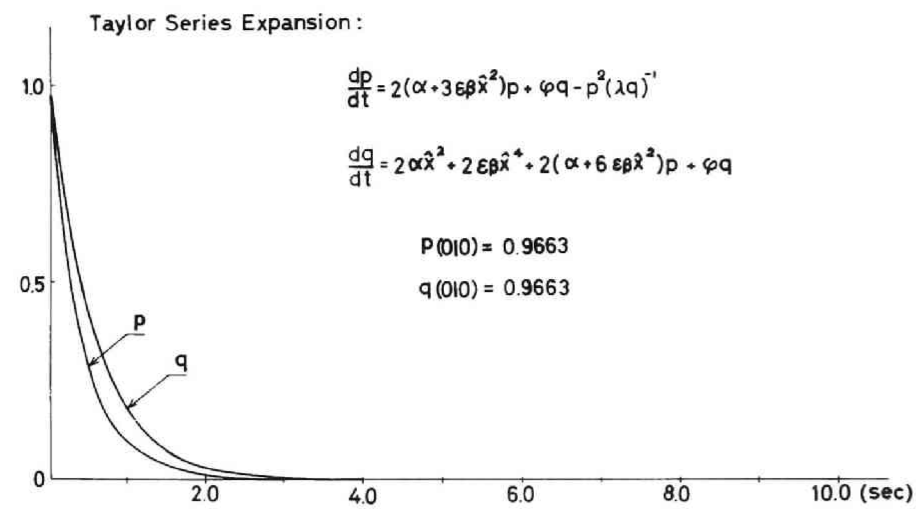


Fig.5.6(b) the output run of the covariance equation

Fig.5.6. A sample path behavior of the approximate filter dynamics (Taylor series expansion) (ε=0.10, ϕ^2=0.20, λ^2=1.00)

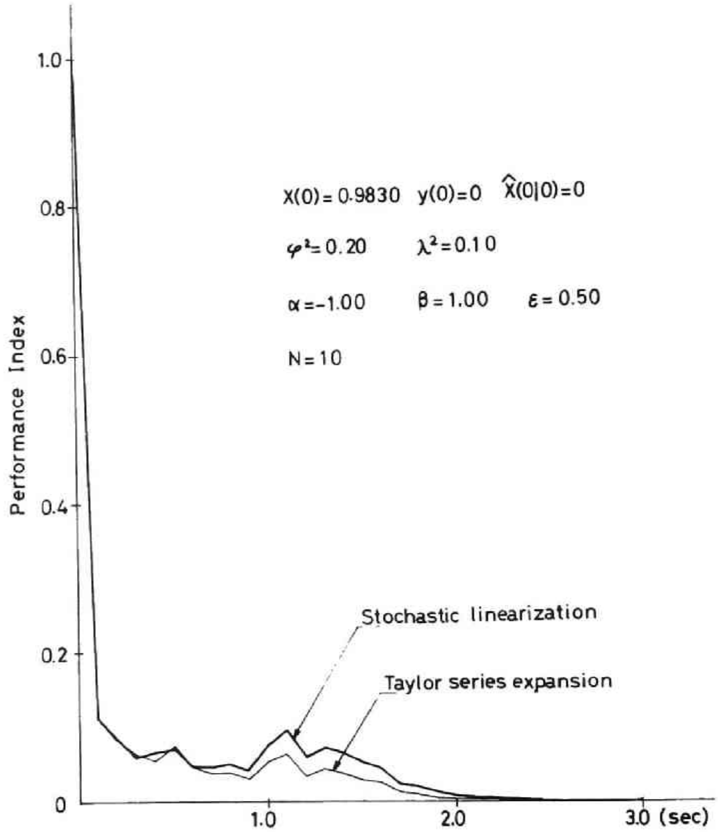


Fig.5.7. Performance evaluation for the two filters.

state and that of the estimate respectively, and N is the number of sample runs to be averaged. Both figures 5.7 and 5.8 show the c(t) run, where the values of parameters correspond to those of figures 5.2, 5.5 and figures 5.3, 5.6. In the figures, the performances $c_S(t)$ of the approximate filter dynamics derived by the stochastic linearization method and $c_T(t)$ of the approximate filter dynamics derived by the Taylor series expansion method are compared with each other. It can be observed that the filter dynamics derived by the Taylor series expansion shows a slightly better performance than that derived by the stochastic linearization technique. However, it should be noted that the stochastic linearization method requires the expansion of a nonlinear function up

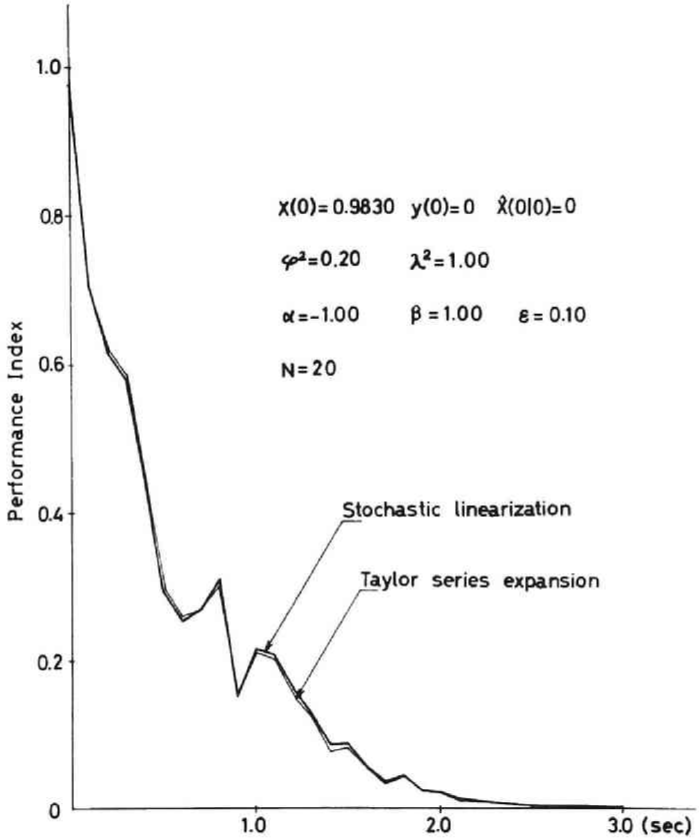


Fig.5.8. Performance evaluation for the two filters.

to the first order of the error $x-\hat{x}$ as shown in (5.3) while the Taylor series method requires the expansion up to the second order of $x-\hat{x}$. Consequently, it may be emphasized that the approximate filter dynamics derived by stochastic linearization method can compete with another filter dynamics through the first order expansion in the system dynamics.

5.5. Performance Evaluation of Approximate Filter Dynamics

This section is concerned with an analytical study on the performance evaluation for the purpose of providing deeper insight into the ramifications of approximation techniques to nonlinear filtering problems.

Concretely speaking, the problem considered here is to evaluate the filtering error defined by $\varepsilon(t)=x(t)-\hat{x}(t|t)$. From practical viewpoints, it is useful to compute the first, the second and more higher order moments of $\varepsilon(t)$. However, the result of theoretical contributions to nonlinear filtering problems reveals that an exact realization of optimal nonlinear filters requires infinite dimensional filters which are practically impossible to realize those[16,78,115,123,158,185]. This implies that the precise evaluation of the filtering error is almost impossible and that both the construction and the related performance evaluation of approximate filter dynamics are highly important. In [7], a trial has been reported on the approximate evaluation of the filtering performance by assuming an approximate filter dynamics. In this section, however, the approximate filter dynamics will be established first and then the approximate evaluation of the filtering error will be performed.

In order to evaluate the filtering error $\varepsilon(t)$, the two approximate estimation processes are considered, which are respectively generated by the filter dynamics derived by (1) the method of stochastic linearization and (2) the method of Taylor series expansion.

5.5.1. System Models and Filter Dynamics

The mathematical models are chosen to be Σ_{1F} defined in Def.2.2, i.e.

$$\left.\begin{aligned} &(5.84) \qquad dx(t) = f[t,x(t)]dt + G(t)dw(t), \\ &(5.85) \qquad dy(t) = h[t,x(t)]dt + R(t)dv(t). \end{aligned}\right\} : \Sigma_{1F}$$

Throughout this section, two Greek letters α and β as subscripts in vectors or matrices are used to distinguish them from other vector and matrices of the same genre. For example, $\hat{x}_\alpha(t|t)$ and $P_\alpha(t|t)$ are the approximate estimation process and the associated error covariance matrix derived by the method of stochastic linearization technique and these symbols are used to distinguish from the true estimation process $\hat{x}(t|t)$ and the error covariance $P(t|t)$. On the other hand, $\hat{x}_\beta(t|t)$ and $P_\beta(t|t)$ are respectively the same quantities as mentioned above but derived by the method of Taylor expansion.

The method of stochastic linearization technique used in the previous sections is also introduced for the purpose of deriving the approximate filter dynamics. Expand the nonlinear function $f(t,x)$ into

$$(5.86)\qquad f[t,x(t)] = a(t) + B(t)\{x(t)-\hat{x}_\alpha(t|t)\} + e_f(t)$$

where $e_f(t)$ denotes the collection of n-dimensional vector error terms, and where $a(t)$ and $B(t)$ are respectively the coefficients of expansion determined under the criterion, $\min_{a(t),B(t)} E\{\|e_f(t)\|^2|Y_t\}$. These coefficients are respectively given by (see Sec.3.2)

$$(5.87a)\qquad a(t) = E\{f[t,x(t)]|Y_t\} \triangleq \hat{f}[t,x(t)]$$

$$(5.87b)\qquad B(t) = E\{[f(t,x)-\hat{f}(t,x)](x-\hat{x}_\alpha)'|Y_t\}P^{-1}(t|t),$$

where

$$(5.88)\qquad P(t|t) = \text{cov.}[x(t)|Y_t].$$

Then the sample path $x(t)$ determined by (5.84) may be approximated by

$$(5.89)\qquad dx_\alpha(t) = B(t)x_\alpha(t)dt + \{a(t)-B(t)\hat{x}_\alpha(t|t)\}dt + G(t)dw(t).$$

The same procedure is applicable to the observation process given by (5.85). Through the expansion of the function $h[t,x(t)]$ in the form,

$$(5.90)\qquad h[t,x(t)] = h_1(t) + H_2(t)\{x(t)-\hat{x}_\alpha(t|t)\} + e_h(t),$$

the coefficients are determined by

$$(5.91a)\qquad h_1(t) = E\{h[t,x(t)]|Y_t\} \triangleq h[t,x(t)]$$

$$(5.91b)\qquad H_2(t) = E\{[h(t,x)-\hat{h}(t,x)](x-\hat{x}_\alpha)'|Y_t\}P^{-1}(t|t).$$

The quasi-linear stochastic differential associated with (5.85) is

$$(5.92)\qquad dy_\alpha(t) = [h_1(t)+H_2(t)\{x(t)-\hat{x}_\alpha(t|t)\}]dt + R(t)dv(t).$$

As the author pointed out in Sec.3.2, in order to calculate the coefficients $a(t)$, $B(t)$, $h_1(t)$ and $H_2(t)$, the conditional pdf of the $x(t)$-process, $p\{t,x(t)|Y_t\}$, is assumed to be Gaussian with the mean value $\hat{x}_\alpha(t|t)$ and the covariance matrix $P_\alpha(t|t)$. By invoking this assumption, each coefficient listed above may be computed as a function of t, $\hat{x}_\alpha(t|t)$ and $P_\alpha(t|t)$. Consequently such more precise symbols as $a(t,\hat{x}_\alpha,P_\alpha)$, $B(t,\hat{x}_\alpha,P_\alpha)$, etc., should be used. Use of these precise symbols will begin with the next subsection.

Based on (5.89) and (5.92), the approximate filter dynamics is given by

$$(5.93)\qquad d\hat{x}_\alpha(t|t) = \hat{f}[t,x(t)]dt + P_\alpha(t|t)H_2'(t)\{R(t)R'(t)\}^{-1} \times\{dy(t)-\hat{h}[t,x(t)]dt\},$$

where $P_\alpha(t|t)$ is the solution of

$$(5.94)\qquad \frac{dP_\alpha(t|t)}{dt} = B(t)P_\alpha(t|t) + P_\alpha(t|t)B'(t) + G(t)G'(t) - P_\alpha(t|t)H_2'(t)\{R(t)R'(t)\}^{-1}H_2(t)P_\alpha(t|t)$$

with $\hat{x}_\alpha(t_0|t_0)=\hat{x}(t_0|t_0)=E\{x(t_0)\}$ and $P_\alpha(t_0|t_0)=P(t_0|t_0)=\mathrm{cov.}[x(t_0)]$.

5.5.2. Performance Evaluation of the Filter Dynamics

The aim of this subsection is to investigate the possibilities and ramifications of obtaining a useful analytical method for evaluating the performance of the approximate filter. To pose the problem for analysis, , equation (5.84) is rewritten by combining it with (5.87),

$$(5.95)\qquad dx(t) = [a(t,\hat{x}_\alpha,P_\alpha)+B(t,\hat{x}_\alpha,P_\alpha)\{x(t)-\hat{x}_\alpha(t|t)\}+e_f(t)]dt + G(t)dw(t).$$

The error process $\varepsilon(t)$ for the filtering process $\hat{x}_\alpha$ is defined by a usual way:

$$(5.96)\qquad \varepsilon(t) = x(t) - \hat{x}_\alpha(t|t),$$

where $\varepsilon(t)$ is an n-vector. Combining (5.93) with (5.95), it follows that

$$(5.97)\qquad d\varepsilon = [a(t,\hat{x}_\alpha,P_\alpha)+B(t,\hat{x}_\alpha,P_\alpha)\varepsilon+e_f-\hat{f}(t,x)]dt + Gdw - P_\alpha H_2'(t,\hat{x}_\alpha,P_\alpha)(RR')^{-1}\{dy-\hat{h}(t,x)dt\}.$$

The innovation process (dy-hdt) in (5.97) is expressed by

$$(5.98)\qquad dy - \hat{h}(t,x)dt = [h_1(t,\hat{x}_\alpha,P_\alpha)+H_2(t,\hat{x}_\alpha,P_\alpha)\varepsilon+e_h-\hat{h}(t,x)]dt + Rdv,$$

where the relations (5.85), (5.90) and (5.96) have been used. Substituting (5.98) into (5.97), we have

$$(5.99)\qquad d\varepsilon = [B-P_\alpha H_2'(RR')^{-1}H_2]\varepsilon dt + (a+e_f-\hat{f})dt -$$

$$- P_\alpha H_2'(RR')^{-1}(h_1+e_h-\hat{h})dt + Gdw - P_\alpha H_2'(RR')^{-1}Rdv.$$

Bearing in mind the relations (5.87a), (5.87b), (5.91a), (5.91b) and the fact that the terms e_f and e_h in (5.86) and (5.90) are of $O(\varepsilon^2)$ respectively, equation (5.99) is approximately expressed by

$$d\varepsilon = L(t,\hat{x}_\alpha,P_\alpha)\varepsilon dt + Gdw - K(t,\hat{x}_\alpha,P_\alpha)dv, \tag{5.100}$$

where

$$L(t,\hat{x}_\alpha,P_\alpha) \triangleq B - P_\alpha H_2'(RR')^{-1}H_2 \tag{5.101}$$

and

$$K(t,\hat{x}_\alpha,P_\alpha) \triangleq P_\alpha H_2'(RR')^{-1}R. \tag{5.102}$$

As the measures of performance evaluation, we shall compute the mean value and covariance of the $\varepsilon(t)$-process, i.e.

$$m(t) \triangleq E\{\varepsilon(t)|x(t_0)=x_0\} \tag{5.103a}$$

$$Q(t) \triangleq \text{cov.}[\varepsilon(t)|x(t_0)=x_0]. \tag{5.103b}$$

From (5.100), it is easily shown that

$$\frac{dm}{dt} = E_0\{L(t,\hat{x}_\alpha,P_\alpha)\varepsilon\}, \tag{5.104}$$

where E_0 is an abbreviated symbol of the conditional expectation $E\{\cdot|x(t_0)=x_0\}$.

Define the $\bar{x}(t)$-process and the covariance matrix by $\bar{x}(t)=E_0\{x(t)\}$ and $\bar{P}(t)=\text{cov.}[x(t)|x(t_0)=x_0]$ respectively. Both the time evolution of $\bar{x}(t)$ and $\bar{P}(t)$ are precisely computed by (5.84), i.e.

$$\frac{d\bar{x}(t)}{dt} = E_0\{f(t,x)\} \triangleq \bar{f}(t,x) \tag{5.105}$$

$$\frac{d\bar{P}(t)}{dt} = E_0\{[f(t,x)-\bar{f}(t,x)](x-\bar{x})'\} + E_0\{(x-\bar{x})[f(t,x)-\bar{f}(t,x)]'\} + G(t)G'(t). \tag{5.106}$$

Instead of the conditional expectation $E\{\cdot|\mathcal{Y}_t\}$ in the relations (5.87a) and (5.87b), if we consider the conditional expectation E_0, then, from (5.105) and (5.106), by a similar method to the stochastic linearization, it is a simple exercise to show that[132]

$$\text{(5.107)} \qquad \frac{d\bar{x}(t)}{dt} = \bar{f}(t,x) \triangleq \bar{a}(t,\bar{x}_\alpha,\bar{P}_\alpha) = \frac{d\bar{x}_\alpha}{dt}$$

$$\text{(5.108)} \qquad \frac{d\bar{P}_\alpha}{dt} = \bar{B}(t,\bar{x}_\alpha,\bar{P}_\alpha)\bar{P}_\alpha + \bar{P}_\alpha\bar{B}'(t,\bar{x}_\alpha,\bar{P}_\alpha) + GG',$$

where

$$\text{(5.109a)} \qquad \bar{x}_\alpha(t) = \bar{x}(t) = E_0\{x(t)\}$$

$$\text{(5.109b)} \qquad \bar{P}_\alpha(t) = \text{cov.}[x(t)|x(t_0)=x_0]$$

and

$$\text{(5.109c)} \qquad \bar{B}(t,\bar{x}_\alpha,\bar{P}_\alpha) = E_0\{(f-\bar{f})(x-\bar{x})'\}\bar{P}_\alpha^{-1}.$$

Noting that $E_0\{\varepsilon(t)\}=E_0[E\{\varepsilon(t)|Y_t\}]$, both the $\bar{x}(t)$-process and $Q(t)$ are respectively observed as the deterministic process. Then expanding the (i,j)-th component of $L(t,\hat{x}_\alpha,P_\alpha)$ in (5.104) into

$$\text{(5.110)} \qquad l_{ij}(t,\hat{x}_\alpha,P_\alpha) = l_{ij}(t,\bar{x},Q) + \sum_{k=1}^{n} \left.\frac{\partial l_{ij}}{\partial \hat{x}_{\alpha k}}\right|_{\bar{x},Q}(\hat{x}_{\alpha k}-\bar{x}_k)$$

$$+ \sum_{k,m=1}^{n} \left.\frac{\partial l_{ij}}{\partial p_{\alpha km}}\right|_{\bar{x},Q}(p_{\alpha km}-q_{km}) + \cdots,$$

where $\hat{x}_{\alpha k}$, $\bar{x}_k$, $p_{\alpha km}$, q_{km} and l_{ij} are components of $\hat{x}_\alpha$, $\bar{x}$, P_α, Q and L, respectively. Deleting the higher-order terms than $O(\varepsilon^2)$ in (5.110), a component-wise expression of (5.104) becomes

$$\text{(5.111)} \qquad \frac{dm_i}{dt} \cong E_0\{\sum_{j=1}^{n}[l_{ij}(t,\bar{x},Q) + \sum_{k=1}^{n}\left.\frac{\partial l_{ij}}{\partial \hat{x}_{\alpha k}}\right|_{\bar{x},Q}(\hat{x}_{\alpha k}-\bar{x}_k)]\varepsilon_j\}.$$

Performing the expectation operation in (5.111), and noting the relation (5.101), it follows that

$$\text{(5.112a)} \qquad \frac{dm}{dt} = L(t,\bar{x},Q)m$$

$$\text{(5.112b)} \qquad = [B(t,\bar{x},Q)-QH_2'(t,\bar{x},Q)(RR')^{-1}H_2(t,\bar{x},Q)]m.$$

On the other hand, from (5.100), (5.103b) and (5.112a), it is easily shown that

$$(5.113)\qquad \frac{dQ}{dt} = E_0\{L(t,\hat{x}_\alpha,P_\alpha)\varepsilon(\varepsilon-m)'+(\varepsilon-m)\varepsilon'L'(t,\hat{x}_\alpha,P_\alpha)$$

$$+ GG' + E_0\{K(t,\hat{x}_\alpha,P_\alpha)K'(t,\hat{x}_\alpha,P_\alpha)\}.$$

Expanding again L and K about $\bar{x}$ and Q, and neglecting the higher-order terms of $O(\varepsilon^2)$ in (5.113), it follows that

$$(5.114)\qquad \frac{dQ}{dt} = L(t,\bar{x},Q)Q + QL'(t,\bar{x},Q) + GG' + K(t,\bar{x},Q)K'(t,\bar{x},Q)$$

$$+ \frac{1}{2} \langle K(t,\bar{x},Q)\partial^2K(t,\bar{x},Q) \ :\ \bar{P}_\alpha-Q\rangle$$

$$+ \frac{1}{2} \langle K(t,\bar{x},Q)\partial^2K(t,\bar{x},Q) \ :\ \bar{P}_\alpha-Q\rangle'$$

$$+ \langle(\partial K(t,\bar{x},Q))^2 \ :\ \bar{P}_\alpha-Q\rangle,$$

where $\langle K(t,\bar{x},Q)\partial^2K(t,\bar{x},Q) \ :\ \bar{P}_\alpha-Q\rangle$ and $\langle(\partial K(t,\bar{x},Q))^2 \ :\ \bar{P}_\alpha-Q\rangle$ are n×n-matrices whose (i,j)-th component are respectively given by

$$(5.115)\qquad \sum_{\nu=1}^{d_2}\sum_{l,m=1}^{n} \frac{\partial^2 k_i}{\partial\hat{x}_{\alpha l}\partial\hat{x}_{\alpha m}}\bigg|_{\bar{x},Q}(\bar{p}_{\alpha lm}-q_{lm})k_{j\nu}(t,\bar{x},Q)$$

and

$$\sum_{\nu=1}^{d_2}\sum_{l,m=1}^{n} \frac{\partial k_{i\nu}}{\partial\hat{x}_{\alpha l}}\bigg|_{\bar{x},Q}\frac{\partial k_{j\nu}}{\partial\hat{x}_{\alpha m}}\bigg|_{\bar{x},Q}(\bar{p}_{\alpha lm}-q_{lm}),$$

and where $\bar{p}_{\alpha lm}$ is an (l,m)-component of the matrix $\bar{P}_\alpha$. In the one-dimensional case, equation (5.114) becomes

$$(5.116)\qquad \frac{dq}{dt} = 2l(t,\bar{x},q)q + g^2 + k^2(t,\bar{x},q)$$

$$+ \left[\frac{\partial^2 k}{\partial\hat{x}_\alpha^2}\bigg|_{\bar{x},q}k(t,\bar{x},q)+\left(\frac{\partial k}{\partial\hat{x}_\alpha}\bigg|_{\bar{x},q}\right)^2\right](\bar{p}_\alpha-q)$$

$$= 2b(t,\bar{x},q)q + g^2 - q^2r^{-2}h_2^2(t,\bar{x},q)$$

$$+ q^2r^{-2}\left[\frac{\partial^2 h_2}{\partial\hat{x}_\alpha^2}\bigg|_{\bar{x},q}h_2(t,\bar{x},q)+\left(\frac{\partial h_2}{\partial\hat{x}_\alpha}\bigg|_{\bar{x},q}\right)^2\right](\bar{p}_\alpha-q).$$

Equations (5.112) and (5.114), or (5.116), are the basic equations for the evaluation of the filter performance. Since $\hat{x}(t_0|t_0)=E\{x(t_0)\}$, the initial condition of (5.112) is $m(t_0)=0$. With this condition, it may

easily be concluded that $m(t)\equiv 0$. The initial value of filtering erro
covariance Q is given by $Q(t_0)=\text{cov.}[\varepsilon(t_0)]$.

So far, the performance evaluation covered up to the second-orde moment. Computations of more higher order moment than the second ord are obviously required in the case of nonlinear filtering problems. Although the same procedure as described in this section is applicabl to evaluate higher order moments, an expected difficulty is tedious calculations. From the viewpoint of practical application, we shall expect to have so many cases where the performance is almost complete by evaluating up to the second-order moment or, at best, up to the third order.

5.5.3. An Illustrative Example with Comparative Discussions

Let us consider a nonlinear dynamical system whose sample process is approximated by a scalar nonlinear stochastic differential equation

$$dx = -\sin x dt + g dw. \tag{5.117}$$

The observation process is simply given by

$$dy = x dt + r dv. \tag{5.118}$$

Based on a couple of equations (5.117) and (5.118), the approxima filter dynamics and the related error variance equation are respective determined by

$$d\hat{x}_\alpha = -\sin\hat{x}_\alpha \exp(-\frac{p_\alpha}{2})dt + p_\alpha r^{-2}(dy-\hat{x}_\alpha dt) \tag{5.119}$$

$$\frac{dp_\alpha}{dt} = -2p_\alpha \cos\hat{x}\exp(-\frac{p_\alpha}{2}) + g^2 - p_\alpha{}^2 r^{-2}. \tag{5.120}$$

The variance equation corresponding to (5.116) becomes

$$\frac{dq}{dt} = -2q\cos\bar{x}\exp(-\frac{q}{2}) + g^2 - q^2 r^{-2}, \tag{5.121}$$

where $\bar{x}$ is a solution of the following differential equation,

---

* It is supposed that the noise level is not so high as to satisfy the existence condition of the solution of (5.117).

$$\frac{d\bar{x}}{dt} = -\sin\bar{x}\,\exp\left(-\frac{\bar{p}_\alpha}{2}\right), \tag{5.122}$$

where $\bar{p}_\alpha$ is given in Table 5.1.

For the purpose of comparative discussions, another approximate filter dynamics is taken into account which is based on the Taylor series expansion for nonlinear function.[111,126] For the approximate filter, the same procedure as mentioned in the preceding section is applicable and somewhat tedious calculations bring

$$\frac{dm}{dt} = \left\{\frac{\partial f}{\partial x}\bigg|_{\bar{x}_\beta} - \left(\frac{\partial h}{\partial x}\bigg|_{\bar{x}_\beta}\right)^2 r^{-2} q + \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}\bigg|_{\bar{x}_\beta} - \frac{\partial h}{\partial x}\bigg|_{\bar{x}_\beta}\frac{\partial^2 h}{\partial x^2}\bigg|_{\bar{x}_\beta} r^{-2} q\right] m\right\} m \tag{5.123}$$

$$\begin{aligned} \frac{dq}{dt} = {} & 2\frac{\partial f}{\partial x}\bigg|_{\bar{x}_\beta} q + g^2 - q^2 r^{-2}\left(\frac{\partial h}{\partial x}\bigg|_{\bar{x}_\beta}\right)^2 \\ & + q^2 r^{-2}\left[\frac{\partial^3 h}{\partial x^3}\bigg|_{\bar{x}_\beta}\frac{\partial h}{\partial x}\bigg|_{\bar{x}_\beta} + \left(\frac{\partial^2 h}{\partial x^2}\bigg|_{\bar{x}_\beta}\right)^2\right](\bar{p}_\beta - q), \end{aligned} \tag{5.124}$$

where $\bar{x}_\beta$ is the solution corresponding to (5.105), i.e.

Table 5.1. Comparison of filter dynamics.

System Dynamics : $dx = -\sin x dt + g dw$

Observation Mechanism : $dy = x dt + r dv$

	Stochastic Linearization	Taylor Series Expansion
Estimation Process	$d\hat{x}_\alpha = -\sin\hat{x}_\alpha \exp(-\frac{p_\alpha}{2})dt + p_\alpha r^{-2}(dy-\hat{x}_\alpha dt)$	$d\hat{x}_\beta = (-\sin\hat{x}_\beta + \frac{1}{2}p_\beta \sin\hat{x}_\beta)dt + p_\beta r^{-2}(dy-\hat{x}_\beta dt)$
Variance Equation	$\frac{dp_\alpha}{dt} = -2p_\alpha \cos\hat{x}_\alpha \exp(-\frac{p_\alpha}{2}) + g^2 - p_\alpha^2 r^{-2}$	$\frac{dp_\beta}{dt} = -2p_\beta \cos\hat{x}_\beta + g^2 - p_\beta^2 r^{-2}$
Equation for Performance Evaluation	$\frac{dq}{dt} = -2q\cos\bar{x}\exp(-\frac{q}{2}) + g^2 - q^2 r^{-2}$ where $\frac{d\bar{x}}{dt} = -\sin\bar{x}\exp(-\frac{\bar{p}_\alpha}{2})$ $\frac{d\bar{p}_\alpha}{dt} = -2\bar{p}_\alpha\cos\bar{x}\exp(-\frac{\bar{p}_\alpha}{2}) + g^2$	$\frac{dq}{dt} = -2q\cos\bar{x}_\beta + g^2 - q^2 r^{-2}$ where $\frac{d\bar{x}_\beta}{dt} = -\sin\bar{x}_\beta + \frac{1}{2}\bar{p}_\beta\sin\bar{x}_\beta$ $\frac{d\bar{p}_\beta}{dt} = -2\bar{p}_\beta\cos\bar{x}_\beta + g^2$

$$(5.125)\qquad \frac{d\bar{x}_\beta}{dt} = f(t,\bar{x}_\beta) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{\bar{x}_\beta}\bar{p}_\beta,$$

and where the one-dimensional case is again considered. Apparently, both equations (5.112) and (5.116) correspond to (5.123) and (5.124) respectively, where the symbols m and q were not distinguished by adding the subscript α and β because no confusion will result. From (5.123) and $m(t_0)=0$, it also follows that $m(t)\equiv 0$.

The results of application of the Taylor series expansion method to (5.117) and (5.118) are listed in Table 5.1.

Comparison of two filter dynamics is found in Table 5.1. Numerical results are shown in Figs.5.9 and 5.10. In these figures, the solid curves depict the q(t)-runs computed respectively by (5.121) and the equation in Table 5.1. Simultaneously, the results of digital simulation are shown by dotts. In Fig.5.1, the dotts were obtained by computing

$$(5.126)\qquad q \cong \frac{1}{N}\sum_{i=1}^{N}(x^{(i)}(t)-\hat{x}_\alpha^{(i)}(t|t))^2,$$

where $x^{(i)}$ and $\hat{x}_\alpha^{(i)}(t|t)$ are respectively the i-th sample process determined by

$$(5.127)\qquad dx^{(i)} = -\sin x^{(i)}dt + gdw$$

and

$$(5.128)\qquad d\hat{x}_\alpha^{(i)} = -\sin\hat{x}_\alpha^{(i)}\exp(-\frac{p_\alpha^{(i)}}{2})dt + p_\alpha^{(i)}r^{-2}\{dy^{(i)}-\hat{x}_\alpha^{(i)}dt\},$$

and where N=100. In Fig.5.10, the dotts were also obtained by replacing $\hat{x}_\alpha^{(i)}$ by $\hat{x}_\beta^{(i)}$ and using Table 5.1. In the example, the system noise and the observation noise variance were $g^2=0.20$ and $r^2=0.10$. The initial value of the state variable was assumed to be Gaussian and that of the estimation was $\hat{x}(t_0|t_0)=0$.

It should be noted that the Taylor series expansion of the nonlinear function requires at least the expansion of up to the second order, while in the case of the stochastic linearization technique, the expansion requires only up to the first order of $\varepsilon=x-\hat{x}$. Then, from the numerical results, it can be said that the approximate filter dynamics derived by

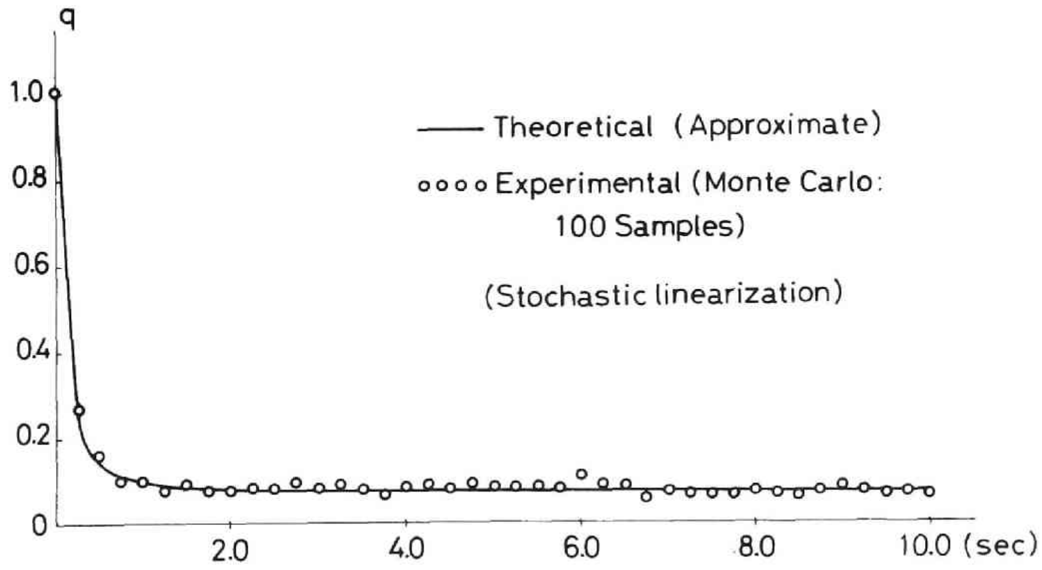


Fig.5.9. Performance evaluation for the stochastic linearization filter.

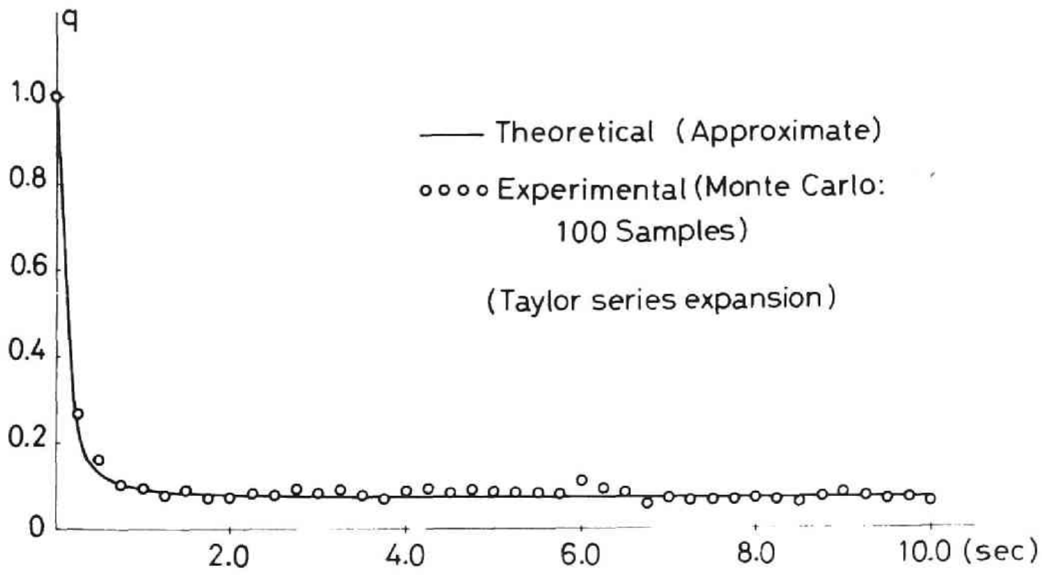


Fig.5.10. Performance evaluation for the Taylor series expansion filter.

the stochastic linearization has a pleasant performance and competes with another approximate filter based on the Taylor series expansion.

5.6. Discussion and Summary

In this section, the approximate filter dynamics has been established for several classes of systems of Σ_{1F}, Σ_{2F} and Σ_{3F} defined in Sec.2.3, Chap.2. Since the state variables are non-Gaussian stochastic processes because of nonlinearities of the system dynamics and of the state-dependent noises for Σ_{2F} and Σ_{3F}, the precise formulation of the optimal filter dynamics becomes also nonlinear. The basic notion of the approximation developed is the linearization technique outlined in Chap.3. In the case where the state-dependent noise is proportional to the system state, the basic notion mentioned above implies that the infinite dimensional filter is approximated by the two-dimensional filter consisting of the first- and second-order moments. However, if the state-dependent noise term is a type of nonlinear function with respect to the system state, then the approximation procedure will become more complicated.

In Sec.5.5, an analytical study of performance evaluation has been developed in order to justify the accuracy of the approximate filter dynamics. With the help of numerical studies, it can be observed that the approximate filter dynamics derived by the stochastic linearization method shows a pleasant performance in comparison with another approximate filter based on the Taylor series expansion.

In the following chapter, it will be shown that the approximate filter derived in this chapter plays a useful role to an extensive application to the scheme of estimation-control for nonlinear stochastic systems.

CHAPTER 6. OPTIMAL STOCHASTIC CONTROL FOR NONLINEAR SYSTEMS UNDER NOISY OBSERVATIONS

6.1. Introductory Remarks

During the past decade, the problem of finding the optimal control has received a great deal of interests as results of the ever-complicated demand to controls and ever-increasing complexity of the operation of modern systems. However, most of this work has concentrated on completely linear dynamical systems, neglecting the effects of nonlinear characteristics exhibited in practice.

There is no need to say that dynamical systems to be controlled exhibit various kinds of nonlinear characteristics and may operate in a random environment whose stochastic characteristics undergo drastic changes. Thus, the general problem to be solved is to find the control of a noisy nonlinear dynamical system in some optimal fashon, given only partial and noisy observations of the system state and, possibly, only an incomplete knowledge of the system. Under such conditions as linearity of the dynamical system, noisy observation and performance criterion given by a quadratic cost functional, it has already been shown that the optimal control problem and optimal estimation problem of the system state from the noise-corrupted observations may independently be solved. [40,55,109,160] However, this is not the case in general for the optimal control of nonlinear dynamical systems, and the overall problems of optimal control and estimation must be carried out simultaneously.

Since the establishment of a precise technique for the optimal control of nonlinear stochastic systems is almost impossible, in this chapter the author introduces an approximate method which is shown to play an important role in the realization of a broad class of stochastic optimal control.

As is well-known, the optimal control is, in general, nonlinear for the problem of designing controls of nonlinear systems. An exact solution of the optimal control problem for nonlinear systems requires the formulation of the stochastic Hamilton-Jacobi-Bellman equation---a quasilinear partial differential equation---whose solution is almost unobtainable without any suitable numerical method. Problems of any significant order lead to obviously intractable computational problems.

One approach to solve such an optimal control problem of nonlinear systems will be approximations to nonlinear functions in some sense by a certain equivalent linear ones and developments in the linear-quadratic-Gaussian (LQG) context. The author thus may find a suboptimal control with use of stochastic linearization technique to approximate the system by an equivalent linear system. Then the computational technique is used associated with linear optimal control design, and the computational difficulties which will arise in solving the stochastic Hamilton-Jacobi-Bellman equation are by-passed. The resulting control reveals to be a linear feedback control which is realistic from the viewpoint of application.

In this chapter, the mathematical formulations for the systems Σ_{1C} and Σ_{2C} are developed to the cost functional,

$$J(u) = E\{F[x(T),x^d(T)] + \int_{t_0}^{T} L[t,x(t),u(t)]dt\}, \tag{6.1}$$

which is given in (1.6) (Chap.1, Sec.1.2). The definition of admissible controls is stated in Sec.6.2, and the basic stochastic Hamilton-Jacobi-Bellman equation is derived for the functional (6.1) in Sec.6.3. Sections 6.4 and 6.5 are devoted to obtain suboptimal controls by an admittedly heuristic approach for nonlinear systems with state-independent noise and with state-dependent noise respectively. Some aspects are considered in Sec.6.6 for numerical computations of suboptimal controls with illustrative examples. In the final section, the prevalence of stochastic linearization technique is emphasized from the viewpoint of the computer-oriented optimal estimation-control systems.

6.2. Definition of Admissible Controls[130]

In this section, let us consider the system Σ_0 defined in Def.2.1 (Chap.2, Sec.2.3):

$$\left.\begin{array}{ll}(6.2) & dx(t) = f[t,x(t)]dt + c[t,u(t)]dt + G[t,x(t)]dw(t), \\ (6.3) & dy(t) = h[t,x(t)]dt + R[t,x(t)]dv(t),\end{array}\right\} : \Sigma_0$$

where $t\varepsilon[t_0,T]$.

Following [160], we proceed to establish the solution of the stochastic differential equations (6.2) and (6.3).

Let G denote the class of continuous functions $g(t)$ defined on $[t_0,T]$ with values in $E^{(n)}$, and F_t denote a functional operator in $E^{(n)}$. Clearly, if $g\varepsilon G$, then $F_t g\varepsilon G$. Furthermore, let ψ denote a mapping of $[t_0,T]\times G$ onto U with the following properties:

(P6.1) For each $g\varepsilon G$, the functional $\psi(t,g)$ is Hölder continuous in t (exponent α), i.e.

$$(6.4) \qquad \|\psi(t,g)-\psi(s,g)\| \leq K_0\|t-s\|^{\alpha}, \quad t,s\varepsilon[t_0,T].$$

(P6.2) For $t\varepsilon[t_0,T]$, the functional ψ satisfies a uniform Lipschitz condition

$$(6.5) \qquad \|\psi(t,g_1)-\psi(t,g_2)\| \leq K_1\|g_1-g_2\|_{sup},$$

where the functions $g_1,g_2\varepsilon G$ and K_0,K_1 are real positive constants, and where $\|\cdot\|_{sup}$ expresses sup norm in G.

Let $\psi(t,\cdot)$ be an m-dimensional vector stochastic process, such that for each $t\varepsilon[t_0,T]$, $\psi(t,\cdot)$ is measurable and

$$\int_{t_0}^{T} E\{\|\psi(t,\cdot)\|^2\}dt < \infty, \tag{6.6}$$

where $\|\cdot\|$ expresses the norm in $E^{(m)}$. Let Ψ be the class of the $\psi(t,\cdot)$-process. For some $\psi\varepsilon\Psi$, we call $u(t)$ admissible and write $u\varepsilon U$, if $u(t)=\psi(t,\cdot)$, $t\varepsilon[t_0,T]$.

Let ψ be a mapping of $[t_0,T]\times E^{(n)}$ onto U and let $\hat{\Psi}$ be a class of functions $\hat{\psi}$, where $\hat{\psi}$ is Hölder continuous (exponent α) in t and satisfies a uniform Lipschitz condition. We write $u\varepsilon\hat{U}\subset U$, if, for $t\varepsilon[t_0,T]$,

$$u(t) = \hat{\psi}[t,\hat{x}(t|t)] \tag{6.7}$$

for some $\hat{\psi}\varepsilon\hat{\Psi}$. In the case where the system states are corrupted by observation noise, we call the control $u(t)$ given by (6.7) admissible.

With the hypotheses described in Def.2.1 and the additional hypotheses (6.6) made on the control term in (6.2), it has already been verified that (6.2) has exactly a unique continuous solution $x(t)$. A precise interpretation of (6.2) and also (6.3) are respectively given by Itô who writes them as the stochastic integral equation:

$$x(t) = x(t_0) + \int_{t_0}^{t} f[s,x(s)]ds + \int_{t_0}^{t} c[s,u(s)]ds + \int_{t_0}^{t} G[s,x(s)]dw(s) \tag{6.8}$$

and

$$y(t) = y(t_0) + \int_{t_0}^{t} h[s,x(s)]ds + \int_{t_0}^{t} R(s)dv(s). \tag{6.9}$$

6.3. Stochastic Hamilton-Jacobi-Bellman Equation

The problem in this section is to derive the basic stochastic Hamilton-Jacobi-Bellman equation in order to find the optimal control which minimizes the cost functional (6.1). In this section and in the sequel, we shall consider the case where the control term in (6.2) is $c[t,u(t)]\equiv C(t)u(t)$.

Along the line of attack on the linear regulator problem in the case of observation noise free, we suppose that $u(t)=\psi[t,x(t)]$. Bearing this in mind, we proceed to a generalization of the quasi-linear filtering equation derived in Chap.5. The problem is stated as follows: Given that $x(t)$ and $y(t)$ have the stochastic differentials, (6.2) and (6.3) respectively, we derive the stochastic differential of the state estimation $\hat{x}(t|t)=E\{x(t)|\mathcal{Y}_t\}$, for $t\varepsilon[t_0,T]$. This problem is easily reduced to that in the previous chapter. The result becomes

$$(6.10a)\qquad d\hat{x}(t|t) = \hat{f}[t,x(t)]dt + C(t)\psi[t,x(t)]dt + P(t|t)H_2'(t)\{R(t)R'(t)\}^{-1}\{dy(t)-\hat{h}[t,x(t)dt\},$$

$$(6.10b)\qquad \hat{x}(t_0|t_0) = E\{x(t_0)\},$$

where

$$(6.11a)\qquad P(t|t) = \text{cov.}[x(t)|\mathcal{Y}_t]$$

$$(6.11b)\qquad P(t_0|t_0) = \text{cov.}[x(t_0)].$$

Equation (6.10) reveals that the optimal estimator dynamics differs from (5.13) only by the addition of the $\mathcal{Y}_t$-measurable drift term $C(t)\psi[t,x(t)]dt$.

It can easily be shown[31] that the filtering process determined by (6.10a) is a diffusion process with the differential generator,

$$(6.12)\qquad L_f V(t,\nu) = V_t(t,\nu) + \{\hat{f}[t,x]+C(t)\psi[t,x]\}'V_\nu(t,\nu) + \frac{1}{2}\text{tr.}\{\Sigma'(t)V_{\nu\nu}(t,\nu)\Sigma(t)\}$$

whenever V is a function defined and of class $C^{(2)}$ on the state space $E^{(n)}$, where

$$(6.13)\qquad \Sigma(t) \triangleq P(t|t)H_2'(t)\{R(t)R'(t)\}^{-1}R(t).$$

Bearing in mind the estimator dynamics given by (6.10), we shall proceed to obtain the optimal control strategy.

Let the function $F[x(T)]$ in (6.1) be

$$(6.14)\qquad F[x(T)] = \|x(T)\|_F^2,$$

where F is a positive semi-definite, real, constant symmetric matrix. Furthermore, let the function $L[t,x(t),u(t)]$ in (6.1) be

$$(6.15)\qquad L[t,x(t),u(t)] = \|x(t)\|^2_{M(t)} + \lambda\|u(t)\|^2_{N(t)}, \qquad (\lambda>0),$$

where M(t) and N(t) are respectively measurable, locally bounded, positive semi-definite and positive definite symmetric matrices. From (6.1), the control problem becomes the minimization of the functional,

$$(6.16)\qquad J(u) = E\{\|x(T)\|^2_F + \int_{t_0}^T \{\|x(t)\|^2_{M(t)}+\lambda\|u(t)\|^2_{N(t)}\}dt\,|\,x(t_0)=x_0\},$$

with respect to u(t). We shall consider the functional,

$$(6.17)\qquad E\{\|x(T)\|^2_F + \int_t^T\{\|x(s)\|^2_{M(s)}+\lambda\|u(s)\|^2_{N(s)}\}ds\,|\,\mathcal{Y}_t\},$$

for $t_0 \le t \le T$. Let $\hat{\psi}$ be the class of control,

$$(6.18)\qquad u(t) = \hat{\psi}[t,\hat{x}(t|t)]$$

and write

$$(6.19)\qquad V(t,\mathcal{Y}_t) = \min_{\hat{\psi}} E\{\|x(T)\|^2_F + \int_t^T\{\|x(s)\|^2_{M(s)}+\lambda\|\hat{\psi}(s,\hat{x}_s)\|^2_{N(s)}\}ds\,|\,\mathcal{Y}_t\},$$

where $\hat{x}_s=\hat{x}(s|s)$, and where $\{\hat{x}_s\}$ is the process determined by letting $u=\psi=\hat{\psi}$ in (6.10). Since $\hat{x}(s|s)$ is measurable relative to the sample space of $\hat{x}(t|t)$ for $t\le s$, we have[90,137]

$$E\{\phi[\hat{x}(s|s)]\,|\,\mathcal{Y}_t\} = E\{\phi[\hat{x}(s|s)]\,|\,\hat{x}(t|t)=\kappa\},$$

where ϕ is an arbitrary measurable function.

Applying the principle of optimality to (6.19), we have

$$\begin{aligned}(6.20)\qquad V(t,\mathcal{Y}_t) &= \min_{\hat{\psi}} E\{\|x(T)\|^2_F + \int_t^T[\|x(s)\|^2_{M(s)} \\ &\qquad +\lambda\|\hat{\psi}(s,\hat{x}_s)\|^2_{N(s)}]ds\,|\,\hat{x}(t|t)=\kappa\} \\ &\cong \min_{\hat{\psi}} E\{[\|x(t)\|^2_{M(t)} + \lambda\|\hat{\psi}(t,\hat{x}_t)\|^2_{N(t)}]dt \\ &\qquad + E\{\|x(T)\|^2_F + \int_{t+dt}^T[\|x(s)\|^2_{M(s)} \\ &\qquad +\lambda\|\hat{\psi}(s,\hat{x}_s)\|^2_{N(s)}]ds\,|\,\hat{x}(t+dt|t+dt)=\kappa+d\kappa\}\,|\,\hat{x}(t|t)=\kappa\} \\ &= \min_{\hat{\psi}} E\{[\|x(t)\|^2_{M(t)} + \lambda\|\hat{\psi}(t,\hat{x}_t)\|^2_{N(t)}]dt +\end{aligned}$$

$$+ V(t+dt,\kappa+d\kappa)|\hat{x}(t|t)=\kappa\}.$$

Finally, from (6.12) and (6.20), the following functional equation is obtained,

$$(6.21) \qquad -V_t(t,\kappa) = \min_{\hat{\psi}} \{\mathrm{tr.}\{M(t)P(t|t)\} + \kappa'M(t)\kappa + \lambda\hat{\psi}'(t,\kappa)N(t)\hat{\psi}(t,\kappa) + [a(t)+C(t)\hat{\psi}(t,\kappa)]'V_\kappa(t,\kappa) + \frac{1}{2}\mathrm{tr.}\{\Sigma'(t)V_{\kappa\kappa}(t,\kappa)\Sigma(t)\}\}.$$

Performing the minimization of (6.21), we have

$$(6.22) \qquad \hat{\psi}^o(t,\kappa) = -\frac{1}{2\lambda}N^{-1}(t)C'(t)V_\kappa(t,\kappa).$$

Substituting (6.22) into (6.21), we have the stochastic Hamilton-Jacobi-Bellman equation,

$$(6.23) \qquad -V_t(t,\kappa) = \mathrm{tr.}\{M(t)P(t|t)\} + \kappa'M(t)\kappa + a'(t)V_\kappa(t,\kappa) - \frac{1}{4\lambda}V_\kappa'(t,\kappa)C(t)N^{-1}(t)C'(t)V_\kappa(t,\kappa) + \frac{1}{2}\mathrm{tr.}\{\Sigma'(t)V_{\kappa\kappa}(t,\kappa)\Sigma(t)\}.$$

6.4. Suboptimal Control for Nonlinear Systems with State-Independent Noise

In this section, the system Σ_0 (Eqs.(6.2) and (6.3)) is limitted to the system with state-independent noise, Σ_{1C}, defined by Def.2.5 (Sec.2.3, Chap.2), that is, we set $G[t,x(t)]\equiv G(t)$ and $R[t,x(t)]\equiv R(t)$ in (6.2) and (6.3). Then the basic filter equation is given by (5.13) with its associated covariance equation (5.15). For the system Σ_{1C}, the partial differential equation (6.23) still holds.

In order to find a more explicit form for (6.22), we assume that (6.23) has a solution

$$(6.24) \qquad V(t,\kappa) = \kappa'\Pi(t)\kappa + 2\kappa'\alpha(t) + \beta(t),$$

where $\Pi(t)$, $\alpha(t)$ and $\beta(t)$ are determined as the solutions of matrix, vector and scalar differential equations, respectively. Applying (6.24) to (6.22), the optimal control is then

(6.25) $$\hat{\psi}^{o}(t,\kappa) = K^{o}(t)\kappa + r^{o}(t),$$

where

(6.26) $$K^{o}(t) = -\frac{1}{\lambda}N^{-1}(t)C'(t)\Pi(t)$$

and

(6.27) $$r^{o}(t) = -\frac{1}{\lambda}N^{-1}(t)C'(t)\alpha(t).$$

It is a simple exercise to show that, for $t_0 \leq t \leq T$, (see Appendix E)

(6.28) $$\frac{d\Pi(t)}{dt} - \frac{1}{\lambda}\Pi(t)C(t)N^{-1}(t)C'(t)\Pi(t) + M(t) = 0,$$

(6.29) $$\frac{d\alpha(t)}{dt} - \frac{1}{\lambda}\Pi(t)C(t)N^{-1}(t)C'(t)\alpha(t) + \Pi(t)a(t) = 0,$$

and that, for $t_0 \leq t \leq T$, $\beta(t)$ satisfies

(6.30) $$\frac{d\beta(t)}{dt} - \frac{1}{\lambda}\alpha'(t)C(t)N^{-1}(t)C'(t)\alpha(t) + 2\alpha'(t)a(t)$$
$$+ \text{tr.}\{M(t)P(t|t)\} + \text{tr.}\{\Sigma'(t)\Pi(t)\Sigma(t)\} = 0.$$

Since the minimal cost functional $V(t,\kappa)$ must satisfy the terminal condition,

(6.31) $$V(T,\kappa_T) = E\{\|x(T)\|_F^2 | \hat{x}(T|T)=\kappa_T\} = \kappa_T'F\kappa_T + \text{tr.}\{P(T|T)F\},$$

the solutions $\Pi(t)$, $\alpha(t)$ and $\beta(t)$ should satisfy the following condition, respectively,

(6.32) $$\Pi(T) = F,\ \alpha(T) = 0 \text{ and } \beta(T) = \text{tr.}\{P(T|T)F\}.$$

In (6.28) and (6.29), both $\Pi(t)$ and $\alpha(t)$ are actually independent of the dynamical characteristics of an observation mechanism, $h(t,x)$ and $R(t)$. The optimal control depends on the cost rate function matrices F, M and N and on the system dynamics $f(t,x)$. An overall configuration is schematically shown in Fig.6.1, in a form of computer-aided feedback control systems. However, a serious difficulty arises in the version of numerical computation on (6.28), (6.29) and (6.30) with (6.32). In fact, the computation of (6.25) with (6.26) and (6.27) has to start with the preassigned initial values of the state estimation $\hat{x}(t_0|t_0)$ and error

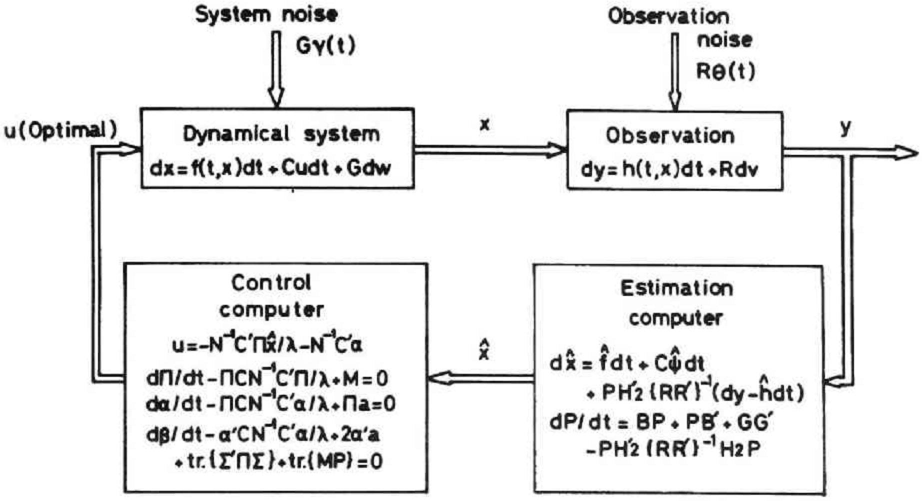


Fig.6.1. Overall configuration of optimal control for nonlinear dynamical systems under noisy observations.

covariance $P(t_0|t_0)$ and, furthermore, with $\Pi(t_0)$ and $\alpha(t_0)$ which are determined by the so-called trial-and-error method or by an improved method stated later in Sec.6.6.

Before stating the method of numerical computations of the optimal control, we establish the control scheme for another system Σ_{2C} defined in Def.2.5 (nonlinear system with state-dependent noise) in the following section.

6.5. Suboptimal Control for Nonlinear Systems with State-Dependent Noise

In this section, the system Σ_{2C} (Def.2.5) is considered and the mathematical development follows on the basis of discussions in Sec.5.3, Chap.5.

Adding the control term to (5.50), the approximate filter dynamics is easily generalized as

$$(6.33a)\qquad d\hat{x} = [\hat{f} + \frac{1}{2}\chi\tilde{G}^2\hat{x}]dt + C\hat{\psi}dt + PH_2'(RR')^{-1}(dy-\hat{h}dt)$$

$$(6.33b)\qquad \hat{x}(t_0|t_0) = 0,$$

where the control u(t) is assumed to be an admissible control of the form $u(t)=\hat{\psi}(t,\hat{x})$. The version of dP/dt is the same form as is given by (5.51).

In the present case, the basic process is $\hat{x}(t|t)$ with the stochastic differential (6.33) and the performance index is given by

$$(6.34)\qquad J(u) = E\{\int_{t_0}^{T} [x'(t)M(t)x(t)+u'(t)N(t)u(t)]dt\},$$

becomes minimal, based on the *a priori* probability distribution on $x(t_0)$, where M and N are measurable, locally bounded, positive semi-definite.

For such a basic process, the suboptimal control problem may be found by the method established in the previous section.

The minimal cost functional is given by

$$(6.35)\qquad V(t,\kappa) = \min_{\hat{\psi}} E\{\int_t^T [x_s'M(s)x_s+\hat{\psi}_s'N(s)\hat{\psi}_s]ds\,|\,\hat{x}(t|t)=\kappa\},$$

where $x_s=x(s)$, $\hat{\psi}_s=\hat{\psi}(s,\hat{x})$ and where x_s is the process determined by reviving $u=\hat{\psi}(s,\hat{x})$ in (5.24). Then the basic functional equation becomes

$$(6.36)\qquad -V_t(t,\kappa) = \min_{\hat{\psi}} \{[a + \frac{1}{2}\chi\tilde{G}^2\kappa + C\hat{\psi}]'V_\kappa(t,\kappa)$$
$$+ \frac{1}{2}\mathrm{tr.}\{\Sigma'V_{\kappa\kappa}(t,\kappa)\Sigma\} + \kappa'M\kappa + \hat{\psi}'N\hat{\psi} + \mathrm{tr.}(MP)\},$$

where $\hat{x}(t|t)=\kappa$, and $\Sigma(t)$ is the same as (6.13), and where the subscripts indicate the derivatives. Performing a minimization operation on the right-hand side of (6.36), the following partial differential equation which corresponds to (6.23) is obtained,

$$(6.37)\qquad -V_t(t,\kappa) = [a + \frac{1}{2}\chi\tilde{G}^2\kappa]'V_\kappa(t,\kappa)$$
$$- \frac{1}{4}V_\kappa'(t,\kappa)CN^{-1}C'V_\kappa(t,\kappa) + \frac{1}{2}\mathrm{tr.}\{\Sigma'V_{\kappa\kappa}(t,\kappa)\}$$
$$+ \kappa'M\kappa + \mathrm{tr.}(MP)$$

with the terminal condition,

$$(6.38)\qquad V(T,\kappa_T) = 0.$$

It may be assumed that (6.38) has a solution,

$$(6.39) \qquad V(t,\kappa) = \kappa'\Pi(t)\kappa + 2\kappa'\alpha(t) + \beta(t).$$

Then the optimal control is approximately obtained by

$$(6.40) \qquad \hat{\psi}^o = - N^{-1}C'(\Pi\kappa+\alpha)$$

and this is adopted here as the suboptimal feedback control strategy. Applying (6.39) and (6.40) to (6.38), $\Pi(t)$, $\alpha(t)$ and $\beta(t)$ are respectively the solutions of the following differential equations,

$$(6.41) \qquad \frac{d\Pi}{dt} + \frac{1}{2}\chi[\tilde{G}^{2\prime}\Pi+\Pi\tilde{G}^2] - \Pi CN^{-1}C'\Pi + M = 0, \quad \Pi(T) = 0,$$

$$(6.42) \qquad \frac{d\alpha}{dt} + \frac{1}{2}\chi\tilde{G}^2\alpha - \Pi CN^{-1}C'\alpha + \Pi a = 0, \quad \alpha(T) = 0,$$

$$(6.43) \qquad \frac{d\beta}{dt} - \alpha'CN^{-1}C'\alpha + 2\alpha'a + \mathrm{tr.}(\Sigma'\Pi\Sigma) + \mathrm{tr.}(MP) = 0, \quad \beta(T) = 0.$$

The version of dQ/dt is changed from (5.54) as

$$(6.44) \qquad \frac{dQ}{dt} = \tilde{B}_\chi Q + Q\tilde{B}_\chi' + a\hat{x}' + \hat{x}a' - (B+CN^{-1}C'\Pi)\hat{x}\hat{x}'$$
$$- \hat{x}\hat{x}'(B+CN^{-1}C'\Pi)' + (a-CN^{-1}C'\alpha)\hat{x}'$$
$$+ \hat{x}(a-CN^{-1}C'\alpha)' + G_0G_0' + G[Q].$$

Thus the suboptimal feedback control is obtained by solving (6.33), (6.40), (6.41), (6.42) and (6.43) simultaneously.

6.6. Some Aspects of Numerical Approach

In the sequel, we merely consider the system Σ_{1C} because parallel discussions on Σ_{2C} are possible.

As pointed out at the end of Sec.6.4, a serious difficulty arises in the numerical computations of (6.28), (6.29) and (6.30) with (6.32). Since the solution matrix $\Pi(t)$ may uniquely be obtained with the terminal condition $\Pi(T)=F$, we shall investigate a practical approach to find the solutions $\alpha(t)$ and $\beta(t)$ of (6.29) and (6.30) satisfying their terminal conditions given by (6.32).

In this section, two possible methods of the computation are investigated.

(1) *Method I.* (Trial-and-Error Method)

Since the solution determined by (6.28) is independent of both the estimate $\hat{x}(t|t)$ and $P(t|t)$, $\Pi(t)$ may uniquely be obtained, which satisfies the terminal condition $\Pi(T)=F$, if the parameter matrices $C(t)$, $N(t)$ and F are given. On the other hand, (6.29) and (6.30) contain the expansion coefficient $a(t)$ as a parameter which depends on both $\hat{x}(t|t)$ and $P(t|t)$. Hence, we have to look for the desired initial values $\alpha(t_0)$ and $\beta(t_0)$. Based on the fact that both the state estimate $\hat{x}(t|t)$ and the error covariance $P(t|t)$ rapidly converge to the steady state $\hat{x}^*$ and P^* in almost every case, we solve the equations with the constant term $a^*=a(\hat{x}^*,P^*)$ instead of $a(t)$ in (6.29) and (6.30) in such a way that the solutions satisfy their terminal conditions. Thus we may find the initial values $\bar{\alpha}(t_0)$ and $\bar{\beta}(t_0)$ and use these for starting the on-line computation. Naturally, this procedure may not give the exact values of $\alpha(t_0)$ and $\beta(t_0)$ which we desire. By the trial-and-error method, it is, therefore, necessary to improve the estimate of the initial values of $\alpha(t)$ and $\beta(t)$ around the *a priori* estimates $\bar{\alpha}(t_0)$ and $\bar{\beta}(t_0)$ so as to realize the desired terminal conditions. The numerical procedure stated above makes it thus possible to perform the overall computer-aided computation scheme.

(2) *Method II.* (Improved Method*)

Assume that, at time t, $a(t)=a_t{}^*$ and, for the time interval $[t,T]$, write the following backward equation for (6.29),

$$\text{(6.45a)} \qquad \frac{d\alpha(\tau)}{d\tau} + \frac{1}{\lambda}\Pi(\tau)C(\tau)N^{-1}(\tau)C'(\tau)\alpha(\tau) - \Pi(\tau)a_t{}^* = 0,$$

$$\text{(6.45b)} \qquad \alpha(\tau)\big|_{\tau=0} = 0,$$

where $0\leq\tau\leq T-t$ and $a_t{}^*$ is a constant. Equation (6.45a) may uniquely be solved in such a way that the solution $\alpha(\tau)$ satisfies the condition $\alpha(\tau)|_{\tau=0}=0$. However, the solution $\alpha(\tau)$ makes sense only at $\tau=T-t$, because

* The author thanks Professor T. Ono, University of Osaka Prefecture, for valuable comments on the improved method.

of the substitution of a_t^* for $a(t)$. Thus, we may have a sequence $\{\alpha(\tau)|_{\tau=T-t}\}$ $(t_0 \leq t \leq T)$ which finally gives us the running value of $\alpha(t)$.

Similarly, for $0 \leq \tau \leq T-t$, we write the following backward equation for (6.30),

$$\frac{d\beta(\tau)}{d\tau} + \frac{1}{\lambda}\boldsymbol{\alpha}'(\tau)C(\tau)N^{-1}(\tau)C'(\tau)\boldsymbol{\alpha}(\tau) - 2\alpha'(\tau)a_t^* \tag{6.46a}$$

$$- \mathrm{tr.}\{M(\tau)P_t^*\} - \mathrm{tr.}\{\Sigma^{*\prime}(\tau)\Pi(\tau)\Sigma^*(\tau)\} = 0,$$

$$\beta(\tau)|_{\tau=0} = \mathrm{tr.}\{P_t^*F\}, \tag{6.46b}$$

where, at time t, $P(t|t)=P_t^*$(constant) and, $H_2(t)=H_{2t}^*$(constant) and

$$\Sigma^*(\tau) = P_t^*H_{2t}^{*\prime}\{R(\tau)R'(\tau)\}^{-(1/2)}. \tag{6.47}$$

By solving (6.46a), the running value of $\{\beta(\tau)|_{\tau=T-t}\}$ is obtained. The optimal control is thus given by

$$u^o(t) = \hat{\psi}^o(t,\kappa) \tag{6.48}$$

$$= -\frac{1}{\lambda}N^{-1}(t)C'(t)\Pi(t)\kappa - \frac{1}{\lambda}N^{-1}(t)C'(t)[\boldsymbol{\alpha}(\tau)]_{\tau=T-t}.$$

The above two methods are applied to digital simulation experiments for a few examples in the next section.

6.7. Digital Simulation Studies and Illustrative Examples

In this section, the digital simulation scheme of the overall system shown by Fig.6.1 is illustrated.

We presume that, at discrete time t_j, the observation δy_j can be taken to be

$$\delta y_j \cong y(j+1) - y(j), \tag{6.49}$$

where, here and in the sequel, t_j is simply expressed by $j(j=0,1,2,\cdots)$. The coefficients $a(t)$, $B(t)$, $h_1(t)$ and $H_2(t)$ can also be computed in discrete form, for instance, from (5.4),

$$a(j) = \mathcal{E}\{f[j,x(j)]|\mathcal{Y}(t_j^{(n)})\} \triangleq \hat{f}[j,x(j)], \tag{6.50a}$$

$$B(j) = \mathcal{E}\{[f[j,x(j)]-\hat{f}[j,x(j)]][x(j)-\hat{x}(j|j)]'|\mathcal{Y}(t_j^{(n)})\} \times P^{-1}(j|j), \tag{6.50b}$$

where

$$(6.51)\qquad \hat{x}(j|j) \triangleq E\{x(j)|\mathcal{Y}(t_j^{(n)})\},\quad P(j|j) \triangleq \mathrm{cov.}[x(j)|\mathcal{Y}(t_j^{(n)})].$$

The notation $\mathcal{Y}(t_j^{(n)})$ denotes the smallest σ-algebra relative to which the random variables $\{y(t_j^{(n)}),\ j=0,1,\cdots,j(n);\ t_0\le t_j^{(n)}\le t\}$ are measurable, where $\{y(t_j^{(n)})\}$ are the random variables partitioned from the $y(t)$-process.

The discrete forms of (6.10) and (5.15) are approximately expressed by

$$(6.52)\qquad \hat{x}(j+1|j+1) = \hat{x}(j|j) + \hat{f}[j,x(j)]\delta_j + C(j)\hat{\psi}[j,\hat{x}(j|j)]\delta_j$$
$$+ P(j|j)H_2'(j)\{R(j)R'(j)\}^{-1}\{\delta y_j - \hat{h}[j,x(j)]\delta_j\}$$

$$(6.53)\qquad P(j+1|j+1) = P(j|j) + B(j)P(j|j)\delta_j + P(j|j)B'(j)\delta_j$$
$$+ G(j)G'(j)\delta_j - P(j|j)H_2'(j)\{R(j)R'(j)\}^{-1}$$
$$\times H_2(j)P(j|j)\delta_j,$$

where $\delta_j = t_{j+1} - t_j$ and where δ_j is sufficiently short. By using $\hat{x}(j+1|j+1)$ obtained by (6.52), with the help of (6.25), (6.26) and (6.27), the suboptimal control signal $u^o(j+1)$ is generated by

$$(6.54)\qquad u^o(j+1) = \hat{\psi}^o[j+1,\hat{x}(j+1|j+1)]$$
$$= K^o(j+1)\hat{x}(j+1|j+1) + r^o(j+1),$$

with

$$(6.55a)\qquad K^o(j+1) = -\frac{1}{\lambda}N^{-1}(j+1)C'(j+1)\Pi(j+1),$$

and

$$(6.55b)\qquad r^o(j+1) = -\frac{1}{\lambda}N^{-1}(j+1)C'(j+1)\alpha(j+1),$$

where both $\Pi(j+1)$ and $\alpha(j+1)$ are, respectively, discrete forms of solutions of (6.28) and (6.29).

The generating routine of random number sequence is a combination of a uniform random sequence plus an approximate transformation to a Gaussian random sequence. To compute $G(j)dw(j) \simeq G(j)\{w(j+1)-w(j)\} = G(j)\delta w_j$ in (6.2), we use the Gaussian random number $n_1(j)$ with $N[0,1]$, where

$n_1(j)=\gamma(j)\sqrt{\delta_j}$. Also, for $R(j)dv(j)\cong R(j)\{v(j+1)-v(j)\}=R(j)\delta v_j$ in (6.3), the Gaussian random number $n_2(j)$ with $N[0,1]$ generated by the different population from $n_1(j)$ is used, where $n_2(j)=\theta(j)\sqrt{\delta_j}$. (See Appendix F, for the simulation of the Brownian motion process.) Thus, (6.2) and (6.3) may, respectively, be simulated as

(6.56) $$x(j+1) = x(j) + f[j,x(j)]\delta_j + C(j)u^o(j)\delta_j + G(j)n_1(j)\sqrt{\delta_j}.$$

(6.57) $$y(j+1) = y(j) + h[j,x(j)]\delta_j + R(j)n_2(j)\sqrt{\delta_j}.$$

The computation procedure is thus established as follows, starting with $\hat{x}(0|0)$, $P(0|0)$, $\Pi(0)$ and $\alpha(0)$ as the initial values:

(i) Obtain $a(t)$, $B(t)$, $h_1(t)$ and $H_2(t)$ by the preassigned nonlinear functions $f[t,x(t)]$ and $h[t,x(t)]$, and establish the forms of $a(j)$, $B(j)$, $h_1(j)$ and $H_2(j)$.

(ii) Preassign the sample values $\hat{x}(0|0)$ and $P(0|0)$ as the given initial values. Simultaneously, by trial-and-error method, determine the value of $\Pi(0)$ and $\alpha(0)$ in such a way that the terminal conditions, $\Pi(n)=F$ and $\alpha(n)=0$ are satisfied, where $n=t_n=T$.

(iii) Determine the value of $u^o(t)=\hat{\psi}^o[0,\hat{x}(0|0)]$ by invoking the preassigned value of $N(0)$, $C(0)$, $\hat{x}(0|0)$, $\Pi(0)$ and $\alpha(0)$.

(iv) For a preassigned value of δ_j, by using the values of $a(j)$, $B(j)$, $h_1(j)$, $H_2(j)$, $\hat{x}(j|j)$ and newly observed data, $y(j+1)$, compute the *a posteriori* estimate $\hat{x}(j+1|j+1)$ and the *a posteriori* error covariance $P(j+1|j+1)$ from (6.25) and (6.53).

(v) Compute $\Pi(j+1)$ and $\alpha(j+1)$ and obtain $K^o(j+1)$ and $r^o(j+1)$.

(vi) With the value of $\hat{x}(j+1|j+1)$ obtained in Step (iv) and the values $K^o(j+1)$ and $r^o(j+1)$ obtained in Step (v), determine the suboptimal control $u^o(j+1)=\hat{\psi}^o[j+1,\hat{x}(j+1|j+1)]$ by (6.54).

(vii) By using the values of $\hat{x}(j|j)$ and $P(j|j)$, compute $a(j+1)$, $B(j+1)$, $h_1(j+1)$ and $H_2(j+1)$.

Letting $j=0,1,\cdots$, Steps (iv) to (vii) give a forwardly recurrent algorithm to obtain simultaneously the running estimate $\hat{x}(j|j)$, $P(j|j)$ and the suboptimal control $u^o(j)$ with $\hat{x}(0|0)$, $P(0|0)$, $\Pi(0)$ and $\alpha(0)$ as a set of given initial data.

Illustrative Example-6.1. For the purpose of exploring the quantitative aspects, we consider here the one-dimensional case where the nonlinear dynamical system is given by the following stochastic differential equation,

$$(6.58) \qquad dx = f(x)dt + cudt + gdw$$

with

$$(6.59) \qquad f(x) = -1 + \cos x.$$

The observation process is simply given by

$$(6.60) \qquad dy = xdt + rdv.$$

Application of (5.4a) and (5.4b) to the present case gives

$$(6.61a) \qquad a(t) = -1 + \cos\hat{x}\exp(-\tfrac{p}{2}),$$

$$(6.61b) \qquad b(t) = -\sin\hat{x}\exp(-\tfrac{p}{2}).$$

From (6.10a) and (6.10b), the approximate estimator dynamics and the related error covariance are respectively determined by

$$(6.62) \qquad d\hat{x} = [-1 + \cos\hat{x}\exp(-\tfrac{p}{2})]dt + cu^{o}dt + pr^{-2}(dy-\hat{x}dt),$$

and

$$(6.63) \qquad \frac{dp}{dt} = -2p\sin\hat{x}\exp(-\tfrac{p}{2}) + g^{2} - p^{2}r^{-2}.$$

Letting m=0 in (6.16), the optimal control is, then, given by

$$(6.64) \qquad u^{o}(t) = \hat{\psi}^{o}(t,\kappa) = k^{o}(t)\kappa + r^{o}(t),$$

with

$$(6.65a) \qquad k^{o}(t) = -\frac{1}{\lambda n}c\pi(t),$$

and

$$(6.65b) \qquad r^{o}(t) = -\frac{1}{\lambda n}c\alpha(t),$$

where $\pi(t)$, $\alpha(t)$ are the solutions of the following differential equations:

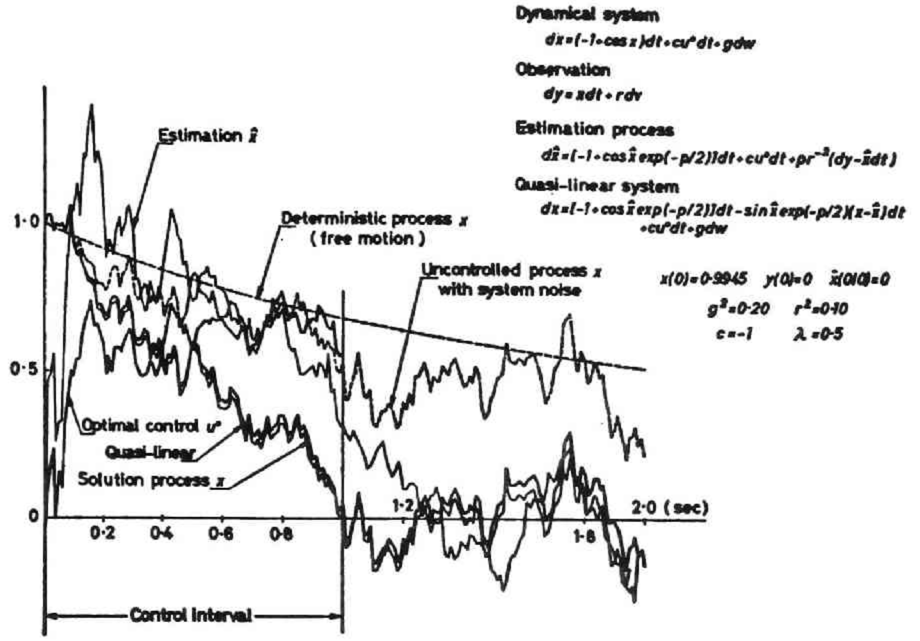


Fig.6.2(a). Sample path behaviors of the system, quasi-linearized system, estimation and optimal control (F=0.5).

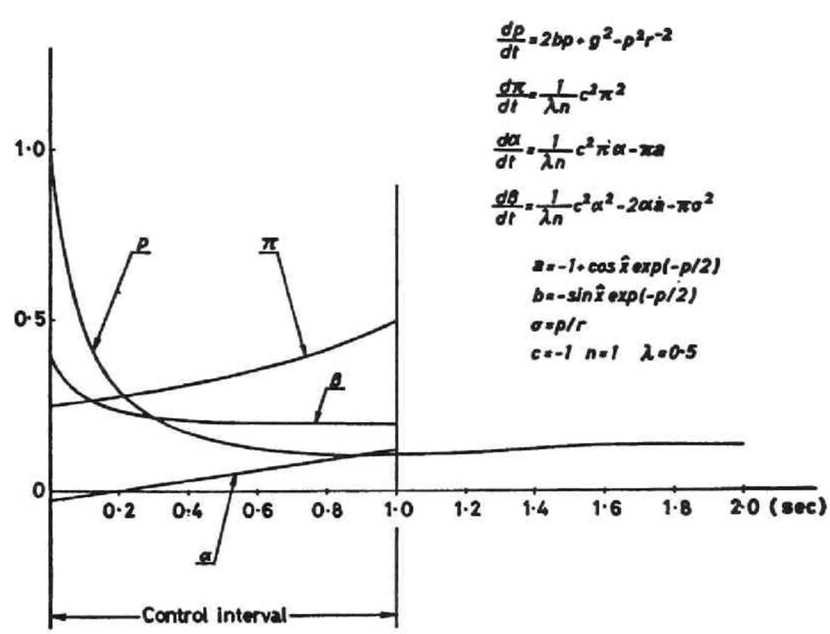


Fig.6.2(b). $p(t|t)$, $\pi(t)$, $\alpha(t)$ and $\beta(t)$ runs.

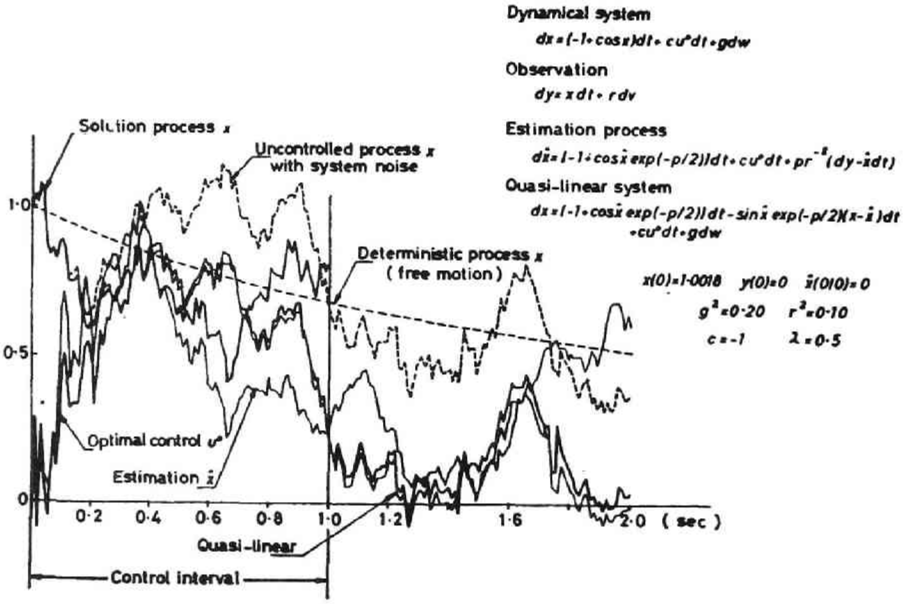


Fig.6.3(a). Sample path behaviors of the system, quasi-linearized system, estimation and optimal control (F=1.0).

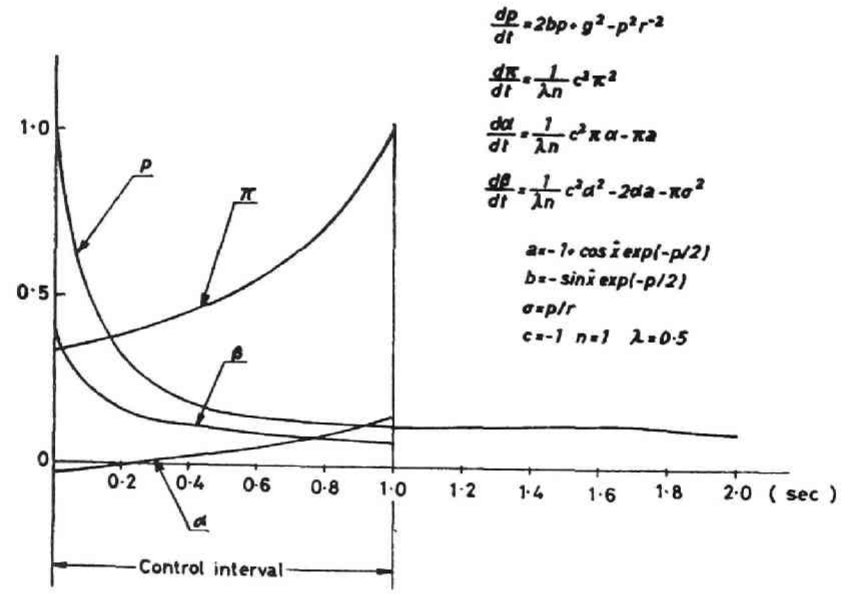


Fig.6.3(b). $p(t|t)$, $\pi(t)$, $\alpha(t)$ and $\beta(t)$.

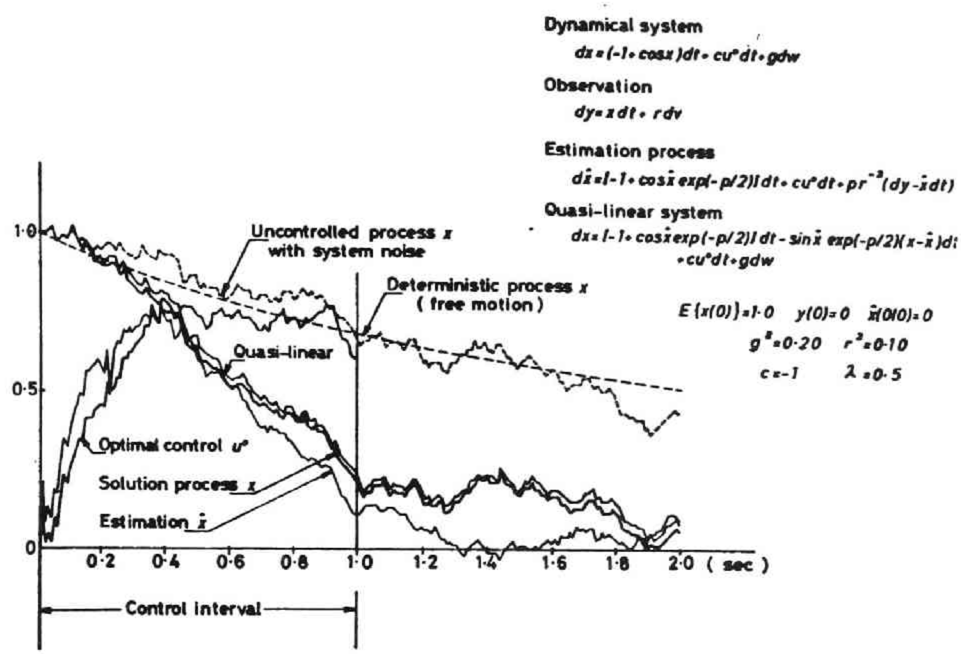


Fig.6.4(a). Averaged runs of ten sample paths (F=1.06).

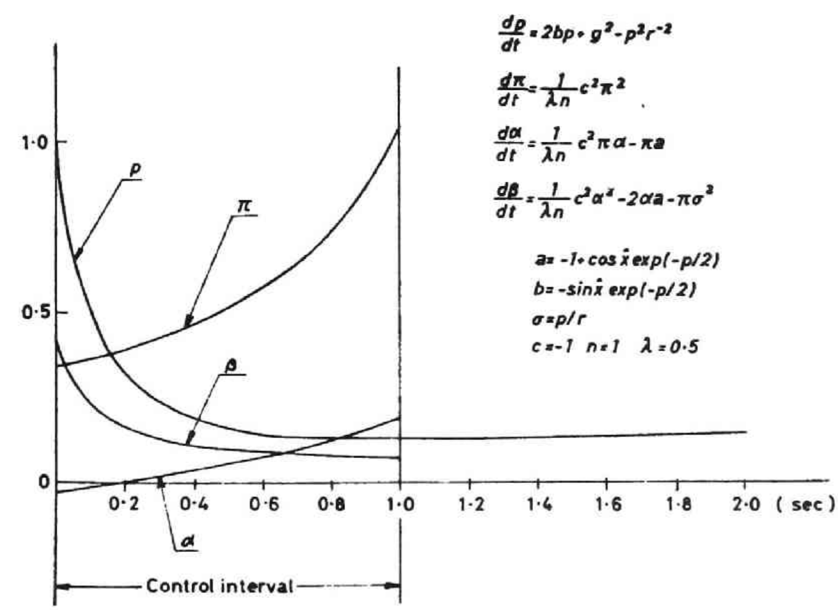


Fig.6.4(b). Averaged runs of ten sample paths of $p(t|t)$, $\pi(t)$, $\alpha(t)$ and $\beta(t)$ processes.

$$\frac{d\pi(t)}{dt} = \frac{1}{\lambda n}c^2\pi^2(t), \tag{6.66}$$

$$\frac{d\alpha(t)}{dt} = \frac{1}{\lambda n}c^2\pi(t)\alpha(t) - \pi(t)a(t). \tag{6.67}$$

Furthermore, the scalar $\beta(t)$ is the solution of the differential equation,

$$\frac{d\beta(t)}{dt} = \frac{1}{\lambda n}c^2\alpha^2(t) - 2\alpha(t)a(t) - \sigma^2(t)\pi(t), \tag{6.68}$$

where

$$\sigma(t) = \frac{1}{r}p(t|t). \tag{6.69}$$

Equations (6.58) to (6.69) are simulated on a digital computer with the subroutine for the generation of random disturbances, $\gamma(t)$ and $\theta(t)$, where λ=0.5 and the control interval is preassigned by [0,1.0](sec).

Figure 6.2(a) shows the running values of the state estimation $\hat{x}(t|t)$, the state of the true system $x(t)$ and the quasi-linearized system $x(t)$, where F=0.5 and n=1.0. However, in practice, the $x(t)$-process cannot be observed and this is only for convenience of discussions.

From Fig.6.2(a), we can observe that the sample path of the system state $x(t)$ with $x(0)$=0.9945, subjected to the optimal control, reaches $x(1.0)$=0.0483. Comparison of the sample path of the quasi-linearized system with that of the true system reveals that the stochastic linearization technique presented is a useful tool for approximations to the state estimation and optimal control for nonlinear dynamical systems. The optimal control signal run is also plotted on Fig.6.2(a). Figure 6.2(b) shows the error covariance $p(t|t)$ of the estimating action, and also $\Pi(t)$, $\alpha(t)$ and $\beta(t)$ which may be adopted as a successful set of trial-and-error methods. Figure 6.3 shows the numerical results of digital simulation studies in the case of F=1.0 and n=1.0. Figure 6.4 shows the average run of 10 sample paths in the case of F=1.06 and n=1.0.

Illustrative Example-6.2. In Example 6.2, the different computational method from Example 6.1 is applied to the same system as in Example 6.1. That is the Method II. A variety of single and averaged-out runs was obtained. In all the experiments, the control interval is preassigned by [0,1.0] and $\lambda=0.5$, $f=1$, $m=0$, $n=1$ and $\delta_j=0.001$(sec). Furthermore, the system noise covariance was $g^2=0.2$ and the observation noise covariance $r^2=0.1$. The results presented below are representative of the simulation experiments.

Figure 6.5(a) shows a single run of the state estimate $\hat{x}(t|t)$, the true value of the system state $x(t)$ (the solution process), the quasi-linearized value of the system state, and the optimal control signal $u^o(t)$. The true initial value of the state variable was $x(0)=0.9945$. There is also interest in observing the true run of the system state without the control. It may be observed that, under the criterion adopted with $\lambda=0.5$, the sample path of the system state is transfered from the initial condition $x(0)=0.9945$ to $x(1.0)=0.0826$ by applying the optimal control. Figure 6.5(b) shows sample paths of the solutions of $p(t|t)$, $\pi(t)$, $\alpha(t)$ and $\beta(t)$ equations.

An averaged behavior of 10 runs with random initial conditions is shown by Fig.6.6(a). The initial states were approximately assumed to be Gaussian random variables. The mean value of the initial states was $E\{x(0)\}=0.9948$. Comparison of the averaged run of the true system state with that of the quasi-linearized system reveals that the stochastic linearization technique developed here is a feasible method for approximations to the state estimation and optimal control for nonlinear dynamical systems.

Illustrative Example-6.3. Let us consider the one-dimensional process whose stochastic equation is given by

$$\text{(6.70)} \qquad dx(t) = [f(t,x) + \tfrac{1}{2}\chi g_1{}^2 x]dt + cu(t)dt + g_0 dw_1(t) + g_1 x dw_2(t),$$

where the nonlinear function is represented by*

* Although for such a nonlinear function, the conditions of the existence and the uniqueness of the solution of (6.70) should be checked out, the author formally uses the function in order to show the usefullness of the stochastic linearization technique.

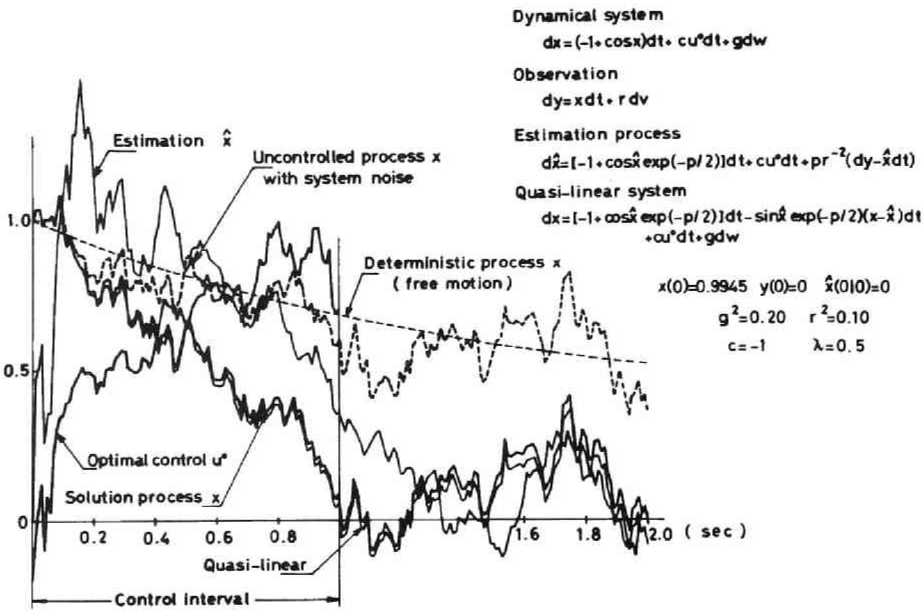


Fig.6.5(a). Sample path behaviors of the system, quasi-linearized system, state estimate and optimal control.

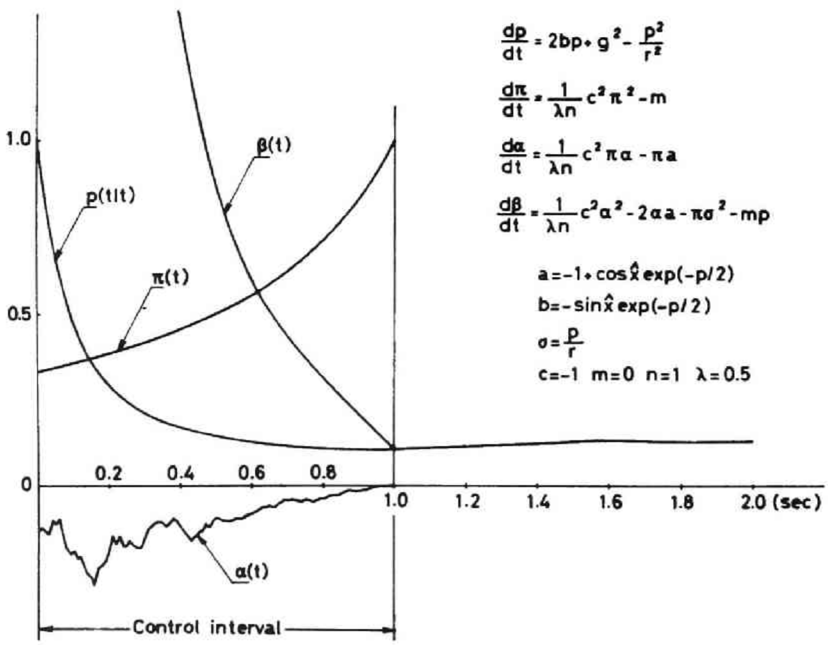


Fig.6.5(b). $p(t|t)$, $\pi(t)$, $\alpha(t)$ and $\beta(t)$ runs.

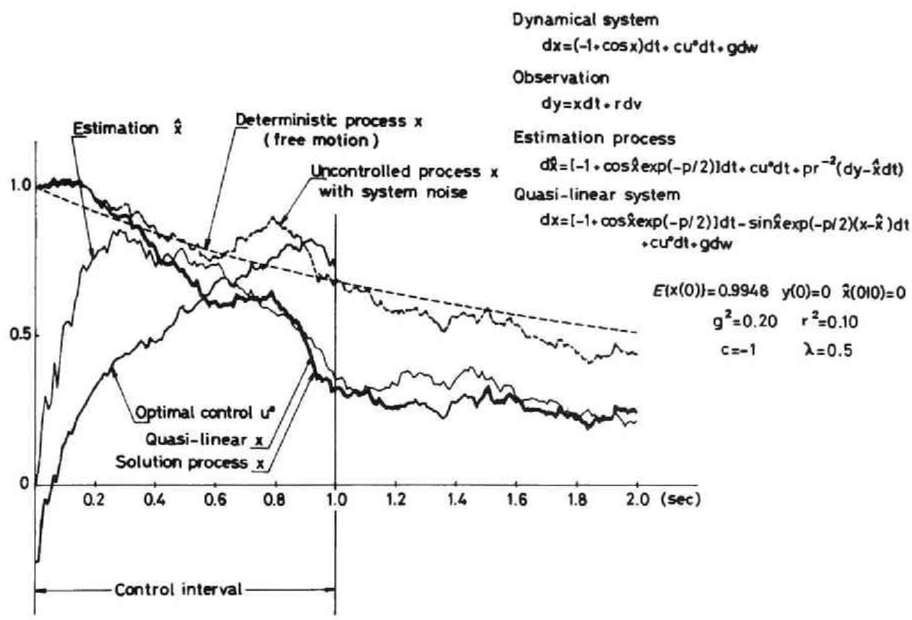


Fig.6.6(a). The averaged performance (10 runs).

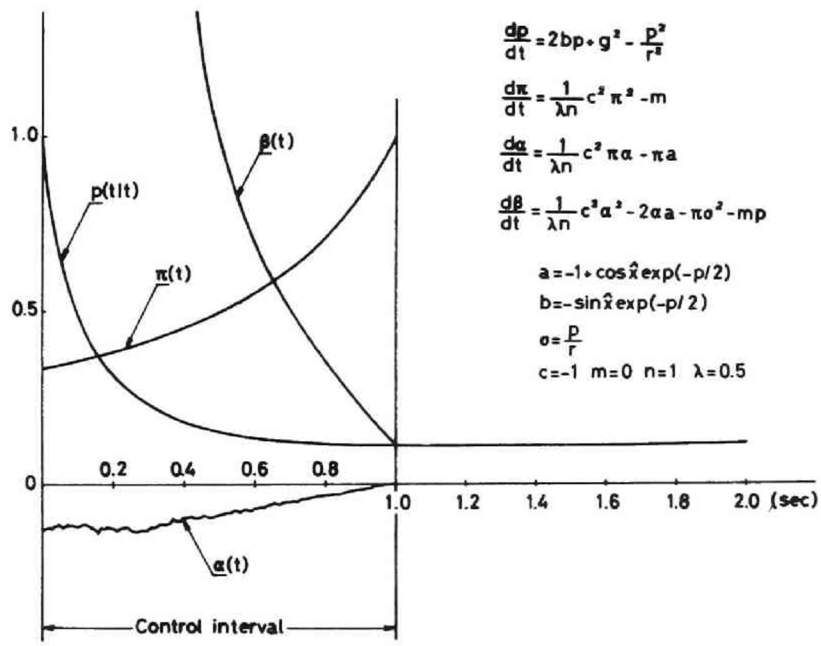


Fig.6.6(b). The averaged runs of $p(t|t)$, $\pi(t)$, $\alpha(t)$ and $\beta(t)$ (10 runs).

$$(6.71)\qquad f(t,x) = \begin{cases} A & \text{for } x>A \\ x & \text{for } |x|\le A \\ -A & \text{for } x<-A. \end{cases}$$

The observation process is simply

$$(6.60)\qquad dy(t) = xdt + rdv(t).$$

Application of (5.4a) and (5.4b) to the present case gives (see Appendix A, Example A.1)

$$(6.72a)\qquad a(t) = \frac{1}{2}[(A+\hat{x})\mathrm{erf}(\frac{A+\hat{x}}{\sqrt{2p}})-(A-\hat{x})\mathrm{erf}(\frac{A-\hat{x}}{\sqrt{2p}})]$$

$$+\sqrt{\frac{p}{2\pi}}[\exp\{-\frac{(A+\hat{x})^2}{2p}\}-\exp\{-\frac{(A-\hat{x})^2}{2p}\}]$$

$$(6.72b)\qquad b(t) = \frac{1}{2}[\mathrm{erf}(\frac{A+\hat{x}}{\sqrt{2p}})+\mathrm{erf}(\frac{A-\hat{x}}{\sqrt{2p}})],$$

where

$$(6.73)\qquad \mathrm{erf}\ \xi = \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-\lambda^2} d\lambda.$$

From (6.33) and (5.51), the approximate filter dynamics and related error covariance are determined by

$$(6.74)\qquad d\hat{x} = [a(t)+\frac{1}{2}\tilde{\chi}g_1^{\,2}\hat{x}]dt + cudt + pr^{-2}\{dy-\hat{x}dt\}$$

$$(6.75a)\qquad \frac{dp}{dt} = 2\tilde{b}_{\chi}p + g_0^{\,2} + g_1^{\,2}q - p^2r^{-2},$$

where

$$(6.75b)\qquad \tilde{b}_{\chi} = b + \frac{1}{2}\tilde{\chi}g_1^{\,2}.$$

The optimal control and the minimal cost functional are given respectively by

$$(6.76)\qquad u^o = -\ cn^{-1}(\pi\hat{x}+\alpha)$$

and

$$(6.77)\qquad V(t,\hat{x}) = \pi\hat{x}^2 + 2\alpha\hat{x} + \beta,$$

where

$$\frac{d\pi}{dt} = -\tilde{\chi g_1}^2\pi + c^2n^{-1}\pi^2 - m, \qquad \pi(T) = 0, \tag{6.78}$$

$$\frac{d\alpha}{dt} = -\frac{1}{2}\tilde{\chi g_1}^2 + c^2n^{-1}\pi\alpha - \pi a, \qquad \alpha(T) = 0, \tag{6.79}$$

$$\frac{d\beta}{dt} = c^2n^{-1}\alpha^2 - 2\alpha a - p^2r^{-2}\pi - mp, \quad \beta(T) = 0. \tag{6.80}$$

The equation corresponding to (6.44) becomes

$$\frac{dq}{dt} = 2\tilde{b}_\chi q + 2a\hat{x} - 2(b+c^2n^{-1}\pi)\hat{x}^2 + 2(a-c^2n^{-1}\alpha)\hat{x} + g_0{}^2 + g_1{}^2q. \tag{6.81}$$

Equations (6.70) to (6.81) are simulated on a digital computer with use of a subroutine for the generation of random disturbances, $w_1(t)$, $w_2(t)$ and $v(t)$. The control interval is preassigned as [0,1.0] (sec). In the simulations, Method II presented in Sec.6.6 was extensively used.

The results of single run experiments are shown by Figs.6.7 and 6.8. Figure 6.7(a) shows five kinds of sample runs obtained by using the mathematical model of the Itô type (χ=0); i.e., the true solution process determined by (6.70), the sample path of a quasi-linear system determined by using (6.72a) and (6.72b), the estimation process $\hat{x}$ by (6.74) and the solution process without a control signal. Naturally, although the true solution process cannot be observed in practice, this is also shown in the figure only for convenience of discussion. Figure 6.7(b) shows the $p(t|t)$, $\pi(t)$, $\alpha(t)$ and $\beta(t)$ runs. In their experiments, the system noise covariances were respectively g_0=0 and $g_1{}^2$=0.4 and the observation noise covariance was r^2=0.1. The true initial value of the state variable was $x(0)$=1.0.

The results of the simulation experiments by using the Stratonovich model are shown by Figs.6.8(a) and 6.8(b) under the same conditions as in Figs.6.7(a) and 6.7(b).

6.8. Prevalence of Stochastic Linearization Technique for the Optimal Stochastic Control

The stochastic linearization technique has been successfully applied in the previous sections to realize the optimal control configuration of

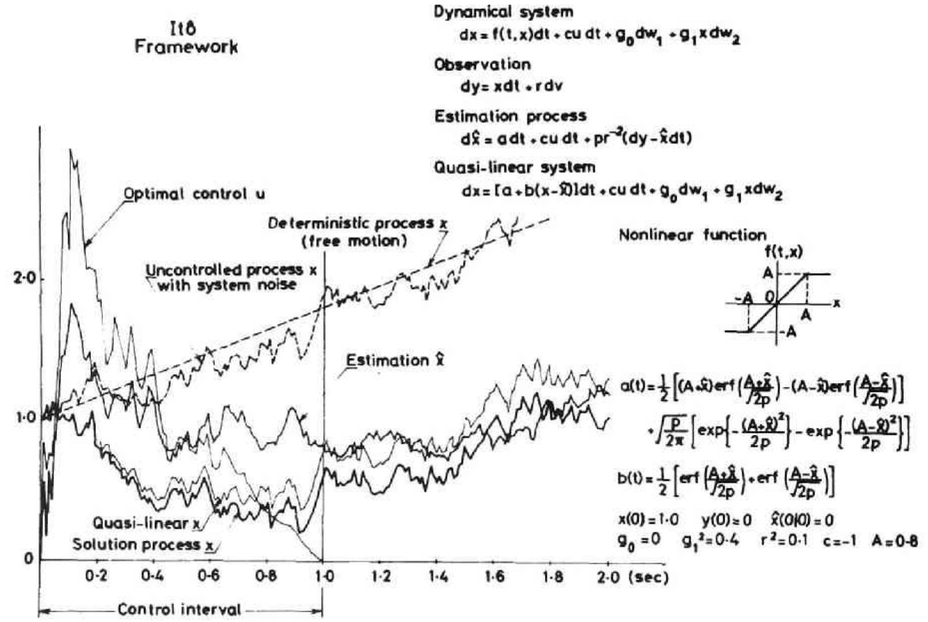


Fig.6.7(a). Sample path behaviors of the system, state estimate and optimal control (Itô framework).

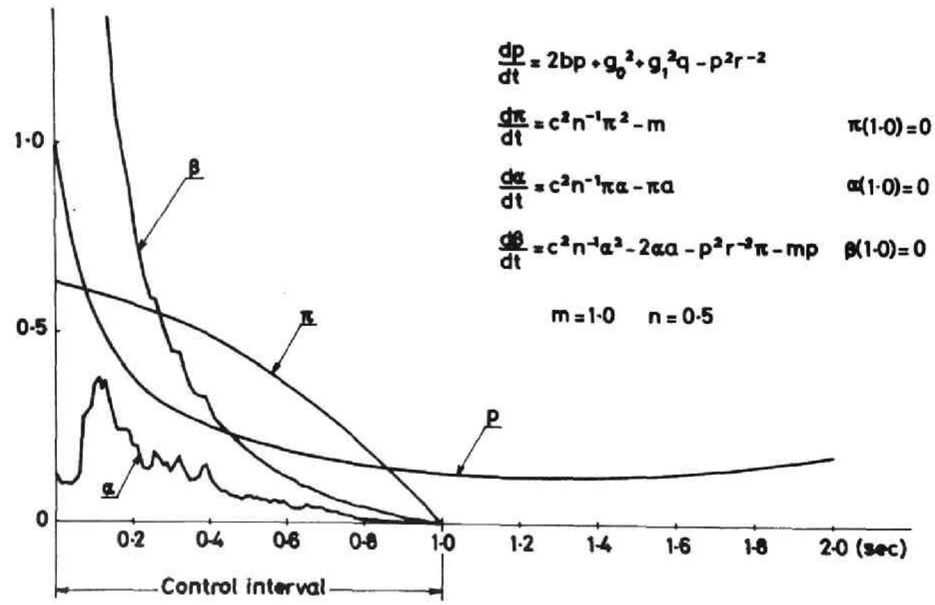


Fig.6.7(b). $p(t|t)$, $\pi(t)$, $\alpha(t)$ and $\beta(t)$ runs.

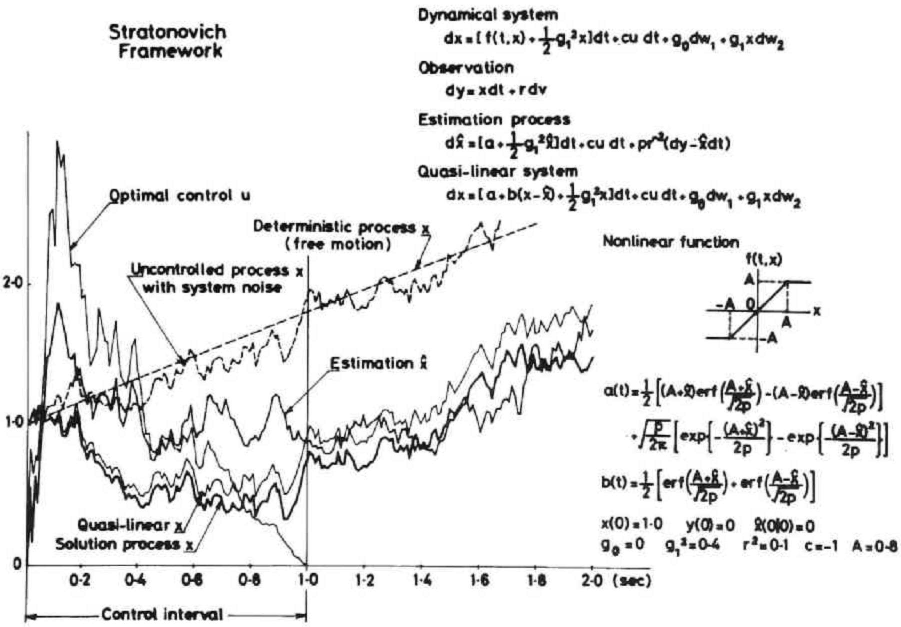


Fig.6.8(a). Sample path behaviors of the system, state estimate and optimal control (Stratonovich framework).

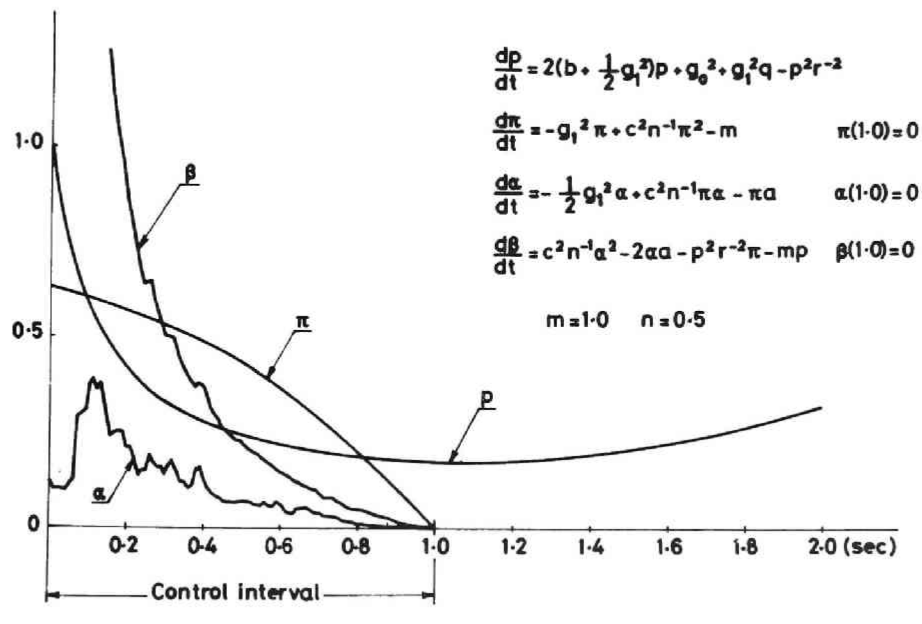


Fig.6.8(b). $p(t|t)$, $\pi(t)$, $\alpha(t)$ and $\beta(t)$ runs.

Table 6.1. Comparison of two methods for estimation-control scheme.

System Dynamics : $dx = f(x)dt + cudt + gdw$

Observation Mechanism : $dy = h(x)dt + rdv$

Cost Functional : $J(u) = E\{\int_{t_0}^{T} [mx^2 + nu^2]dt\}$

	Stochastic Linearization	Taylor Series Expansion
Estimation Process	$d\hat{x}_\alpha = \hat{f}(x)dt + p_\alpha h_2(t)r^{-2}\{dy - \hat{h}(x)dt\}$	$d\hat{x}_\beta = [f(\hat{x}_\beta) + \frac{1}{2}f''(\hat{x}_\beta)p_\beta]dt$ $+ p_\beta h'(\hat{x})r^{-2}\ dy - [h(\hat{x}_\beta) + \frac{1}{2}h''(\hat{x}_\beta)p_\beta]dt$
Covariance Equation (u=0)	$\frac{dp_\alpha}{dt} = 2b(t)p_\alpha + g^2 - p_\alpha{}^2 r^{-2} h_2{}^2(t)$	$dp_\beta = 2f'(\hat{x}_\beta)p_\beta dt - p_\beta{}^2 r^{-2} h'(\hat{x}_\beta)^2 dt + g^2 dt$ $- \frac{1}{2}p_\beta{}^2 r^{-2} h''(\hat{x}_\beta)\{dy - [h(\hat{x}_\beta) + \frac{1}{2}h''(\hat{x}_\beta)p_\beta]dt\}$
Basic Functional Equation ($\hat{x}=\kappa$)	$V_t = a(t)V_\kappa - \frac{1}{4}c^2 n^{-1} V_\kappa{}^2 + \frac{1}{2}{}^2(t)V_{\kappa\kappa}$ $+ m\kappa^2 + mp_\alpha$	$V_t = [f(\kappa) + \frac{1}{2}p_\beta f''(\kappa)]V_\kappa - \frac{1}{4}c^2 n^{-1} V_\kappa{}^2$ $+ \frac{1}{2}\sigma^2(t)V_{\kappa\kappa} + m\kappa^2 + mp_\beta$
Terminal Condition	$V(T,\kappa_T) = 0$	$V(T,\kappa_T) = 0$
Solution (Assumed)	$V(t,\kappa) = \pi(t)\kappa^2 + 2\alpha(t) \ + \beta(t)$	unknown at present
Optimal Stochastic Control	$u^o(t) = \hat{\varphi}^o(t,\kappa)$ $= n^{-1}c[\pi(t)\kappa + \alpha(t)]$	unknown at present

nonlinear systems under noisy observations and a feasible tandem form of optimal estimation-control system has been established.

The key notion of the estimation-control in Secs.6.4 and 6.5 is obviously the stochastic linearization based on the first-order approximation and the assumption of quadratic solution for the basic equation. On the other hand, for the Taylor series expansion filter, however, the basic functional equation contains the nonlinear function in itself as shown in Table 6.1 (where the one-dimensional case is considered, and the subscripts α and β in $\hat{x}$ and p denote the approximated processes of estimation and covariance derived by the two methods of stochastic linearization and Taylor series expansion respectively as used in Sec. 5.5, Chap.5), and therefore such a quadratic solution is extremely difficult to be assumed for the basic functional equation. Since the analytical solution is unobtainable, the avenue to success for estimation-control scheme is almost despairingly closed. Table 6.1 shows the possibilities of both state estimation scheme and control algorithm for each approximation method (Stochastic linearization and Taylor series expansion). According to the inspection of Table 6.1, it should be emphasized that our stochastic linearization technique is the most powerful tool and plays a useful role in the version of state estimation and optimal control problems.

6.9. Discussions and Summary

In this chapter, based on the definition of admissible controls defined in Sec.6.2, the stochastic Hamilton-Jacobi-Bellman equation was derived by using the dynamic programming approach to the quadratic cost functional in Sec.6.3. In Secs.6.4 and 6.5, possible solutions were shown to the stochastic Hamilton-Jacobi-Bellman equation for systems with state-independent and/or state-dependent noises, and then a practical method of estimation-control scheme was proposed in a form of computer-oriented control systems. In Sec.6.6, some aspects of numerical approaches for estimation-control systems were stated, and in Sec.6.7, the method of digital simulation studies was presented with a few illustrative examples. In Sec.6.8, the prevalence of the stochastic linearization technique was emphasized.

It was found that both state estimation and control scheme were facilitated by introducing the stochastic linearization technique and that the joint method of estimation-control was easily implemented by digital computers. Many problems remain ahead. In particular, it was not yet been possible to demonstrate under what conditions a unique solution exists to the optimization problem. The general question is very difficult and this is of more than purely mathematical interest. Although the author's many computational experiences indicate rapid and near-monotone convergence, nor has it been possible to prove convergence of the proposed algorithm. Finally, although the performance evaluation of the approximate filter was done in Sec.5.5, Chap.5, the accuracy of the estimation-control scheme established is still uncertain because the precise solution to the Hamilton-Jacobi-Bellman equation is almost unobtainable for nonlinear systems. However, the proposed technique offers perhaps the only computationally feasible way of arriving at "good" controls for a broad class of nonlinear control systems under noisy observations.

CHAPTER 7. INFORMATION STATES FOR STOCHASTIC CONTROL SYSTEMS

7.1. Introductory Remarks

In recent years much attention has been paid to the various "information patterns" in the theory of classical or nonclassical stochastic control processes[170-172]. The information pattern represents all information about the past history of the process and is the specification of the data which is available for a future control policy. In general the information pattern increases in size and grows in complexity as time goes on. Therefore when a large amount of data is available for performing the optimal control, it is required to summarize it in such a way that no valuable information is deleted. In the development of the theory of dynamic programming and stochastic control, for the purpose of data reduction the important concept of sufficient statistics was noted by Bellman[9,173,174]. The concept of sufficient statistics was particularly emphasized and developed by Striebel[125] and by Aoki[2], forcing us to look deeper into its mathematical importance in the optimal control of stochastic systems.

Independently, Stratonovich gave the concept of "sufficient coordinate" which is a change in form (applicable to the theory of optimal control) of the sufficient statistics, and investigated it in Ref.[123]. In terms of the "information state," some interesting results of the sufficient statistics were obtained by Bohlin[12] and by Davis and Varaiya[25] for discrete- and continuous-time stochastic systems.

The purpose of this chapter is to find the conditions for the "informative" quantity — an information state — which is equivalent to the observation data up to the present time, all the *a priori* knowledges of the system and the past control in describing the future evolution of the system process. In Sec.7.3, on the analogy of the definition in [25], a definition of the information state is given, and an equivalent information state is defined in Sec.7.4. The condition of information states and some typical information states are presented for adaptive control systems and for systems in signal detection problems respectively in Secs.7.5 and 7.6. Summaries and discussions about the information states are given in the final section for various types of (classical) stochastic control systems.

7.2. Preliminaries

The basic system under consideration is modeled by the Itô stochastic differential equations of the form,

$$(7.1)\quad \Sigma_N: \begin{cases} dx(t) = f[t,x(t)]dt + C(t)u(t)dt + G(t)dw(t) \\ dy(t) = h[t,x(t)]dt + R(t)dv(t), \quad t\varepsilon[0,T], \end{cases}$$

which is the same as Σ_0 defined in Def.2.1 (Chap.2, Sec.2.3), except the assumption $c[t,u(t)]=C(t)u(t)$. The system (7.1) is referred to as Σ_N. In the sequel, instead of Σ_N, some different systems such as the linear system Σ_L, an adaptive system Σ_{AN}, etc. are defined.

Let t be a fixed time, and Y_s^t and U_s^t represent the collections of random variables $\{y(\tau),\ s\leq\tau\leq t\}$ and $\{u(\tau),\ s\leq\tau<t\}$. Furthermore, let I_s be a set of the knowledge on the system at time s, that is $I_s=\{Y_0^s, U_0^s\}$. Particularly I_0 is the *a priori* information and consists of the initial state x_0. The set I_t and also the set which is defined by

(7.2) $\quad I_s^t = \{I_s, Y_s^t, U_s^t\}$

will be called the *information patterns*. Obviously, $I_t = I_s^t$. The information pattern I_t determines a σ-algebra in the probability space,

(7.3) $\quad \mathcal{I}_t = \sigma\{I_s, y(\tau), u(\tau);\ s \leq \tau \leq t\}$.

The σ-algebra $\mathcal{I}_t$ will be called the *information σ-algebra*. If the control $u(t)$ is adapted to $\mathcal{Y}_t$, $u(t)$ is a functional on Y_0^t, and the process $u(t)$ is called a feedback control on observations[175].

7.3. Information State

Let there exist an $\mathcal{I}_t$-measurable function $\alpha_t(\omega)$ having its values in a certain measurable space $(A, \mathcal{A})$, $A \subset R^{(n)}$. For the function α_t, an information state is defined on an analogy in [25] as follows.

Definition 7.1. (Information state for the cond. pdf) A process $\{\alpha_t\}$ is an information state for the conditional probability density function (cond. pdf) $p\{x_t | \mathcal{I}_t\}$ if the following conditions are satisfied for given $\mathcal{I}_t$:

(i) α_t is adapted to $\mathcal{I}_t$,

(ii) the density $p\{x_t | \mathcal{I}_t\}$ depends on the information pattern I_t only through α_t,

and

(iii) α_t can be computed recursively, i.e. for any $s<t$, α_t has the form

$$\alpha_t = F(\alpha_s, I_s^t).$$

Roughly speaking, the information state defined by Def.7.1 is one which carries all the relevant information in the past observations and controls [25]. The condition (ii) states that α_t constitutes a sufficient statistic. The space $(A, \mathcal{A})$ will be referred to as the information state space.

Remark 7.1: If the control U_0^t is generated so as to be $\mathcal{Y}_t$-measurable, then $\mathcal{I}_t$ in Def.7.1 may be replaced by $\mathcal{Y}_t$.

Theorem 7.1. If α_t is an information state for the cond. pdf $p\{x_t | \mathcal{I}_t\}$, then

(7.4) $p\{x_t|\mathcal{I}_t\} = p\{x_t|\alpha_t\}$.

Proof. In order to prove (7.4) it suffices to show $p\{x_t|I_t\}=p\{x_t|\alpha_t\}$ for every $I_t \epsilon \mathcal{I}_t$, where $x_t=x(t)$. By definition of cond. pdf

$$(7.5) \qquad p\{x_t|I_t\} = \frac{p\{x_t,I_t\}}{p\{I_t\}} = \frac{p\{x_t,I_t\}}{\int_{E(n)} p\{x_t,I_t\}dx_t}.$$

If α_t is an information state, that is by (ii) in Def.7.1 α_t is a sufficient statistic, then the joint pdf $p\{x_t,I_t\}$ is factored as

$$(7.6) \qquad p\{x_t,I_t\} = p\{x_t,\alpha_t\}g(I_t),$$

where g is a function of I_t which does not depend on x_t. The relation (7.6) is known as the factorization theorem or the Fisher-Neyman criterion for sufficient statistic (see, e.g. [176,p.101] or [177,pp.355-356]).

Substituting (7.6) into (7.5), we have

$$(7.7) \qquad p\{x_t|I_t\} = \frac{p\{x_t,\alpha_t\}g(I_t)}{\int_{E(n)} p\{x_t,\alpha_t\}g(I_t)dx_t} = \frac{p\{x_t,\alpha_t\}}{\int_{E(n)} p\{x_t,\alpha_t\}dx_t} = p\{x_t|\alpha_t\}.$$

This completes the proof.

Consider a linear stochastic system

$$(7.8) \qquad \Sigma_L: \begin{cases} dx(t) = A(t)x(t)dt + C(t)u(t)dt + G(t)dw(t) \\ dy(t) = H(t)x(t)dt + R(t)dv(t), \end{cases}$$

where u(t) is a feedback control (i.e. u is $\mathcal{Y}_t$-measurable); and A and H are n×n- and m×n-matrices. For the system Σ_L we have the important proposition.

Proposition 7.1. For Σ_L, the optimal estimate $\hat{x}(t|t)=E\{x_t|\mathcal{Y}_t\}$ is an information state for the cond. pdf $p\{x_t|\mathcal{Y}_t\}$; i.e. $\alpha_t=(\hat{x}_t)$.

Proof. Note that $p\{x_t|\mathcal{I}_t\}=p\{x_t|\mathcal{Y}_t \times \mathcal{U}_t\}=p\{x_t|\mathcal{Y}_t\}$ since u is $\mathcal{Y}_t$-measurable and $I_t=\{Y_0{}^t\}$. First, (i) the optimal estimate $\hat{x}_t$ is $\mathcal{Y}_t$-measurable. Secondly, (ii) $\hat{x}_t$ is obviously a sufficient statistic. In fact, for Σ_L

the cond. pdf $p\{x_t|\mathcal{Y}_t\}$ is given by

$$p\{x_t|\mathcal{Y}_t\} = c_t \exp\{-\tfrac{1}{2}\|x_t-\hat{x}_t\|^2_{P^{-1}(t|t)}\},$$

where c_t is a normalizing coefficient and $P(t|t)=\text{cov.}[x_t|\mathcal{Y}_t]$. Write this by $g_0(x_t,\hat{x}_t)$. Then we have a representation for the joint pdf,

$$(7.9)\qquad p\{x_t,Y_0{}^t\} = g_0(x_t,\hat{x}_t)g(Y_0{}^t), \quad Y_0{}^t\varepsilon\mathcal{Y}_t$$

which is just the Fisher-Neyman criterion showing that $\hat{x}_t$ is a sufficient statistic. Finally, (iii) $\hat{x}_t$ is obtained recursively by the well-known Kalman-Bucy filter. (Q.E.D.)

In control problems, the control function $u(t)$ is chosen so as to minimize a cost functional

$$(7.10)\qquad J(u) = E\{\int_0^T L(t,x_t,u_t)dt|x_0\},$$

where L is a positive scalar function.

Let ψ be a mapping of $[0,T]\times A$ onto U with the properties: $\psi(t,\cdot)$ is Hölder continuous in t and satisfies a uniform Lipschitz condition. Then the control $u(t)$ is admissible if $u(t)=\psi(t,\cdot)$ (see, Sec.6.2, Chap.6).

<u>Proposition 7.2</u>. If α_t is an information state for the cond. pdf $p\{x_t|I_t\}$, then the optimal control for (7.10) is the function of α_t, i.e.

$$(7.11)\qquad u^o(t) = \psi(t,\alpha_t).$$

For Σ_L, the optimal control is

$$(7.12)\qquad u^o(t) = \psi(t,\hat{x}_t).$$

Proof. The control function $u(t)$ is defined for all possible values of the given information pattern I_t. Define the minimal cost functional by

$$(7.13)\qquad V(t,I_t) = \min_{u_t} E\{\int_t^T L(s,x_s,u_s)ds|I_t\}.$$

Let $u(t)$ be an arbitrary control such that $u(t)=\psi(t,\cdot)$. Then the minimal cost functional $V(t,I_t)$ becomes

$$(7.14)\qquad V(t,I_t) = \min_{\psi_t} E\{\int_t^T L(s,x_s,\psi_s)ds|I_t\},$$

where $\psi_t=\psi(t,\cdot)$. By Theorem 7.1 (7.14) yields the dynamic programming equation in information state space,

$$(7.15) \qquad V(t,I_t) = \min_{\psi_t} E\{\int_t^T L(s,x_s,\psi_s)ds|\alpha_t\}$$

$$= \min_{\psi_t} [E\{\int_t^{t+dt} L(s,x_s,\psi_s)ds|\alpha_t\} + V(t+dt,I_{t+dt})],$$

which gives the optimal control $u^o(t)$ as a function of α_t [90,p.343],

$$(7.16) \qquad u^o(t) = \psi(t,\alpha_t).$$

The second assertion follows by noting that for Σ_L $\alpha_t=(\hat{x}_t)$ by Proposition 7.1. (Q.E.D.)

Remark 7.2: The equality (7.16) shows the separation theorem which was proved by Wonham[160].

7.4. Equivalent Information State

In this section a new concept of the "equivalent information state" is introduced. As ever seen in many stochastic control problems, the *a posteriori* pdf of the system state x(t), i.e. $p\{x_t|\mathcal{I}_t\}$, plays an important role for calculating the optimal estimate and/or control. In Sec.7.3, it was shown that for the linear system Σ_L the optimal estimate $\hat{x}_t$ is an information state for the cond. pdf $p\{x_t|\mathcal{I}_t\}$ (Proposition 7.1). Based on this fact, one can say from a somewhat different viewpoint that the cond. pdf itself is *equivalent* to $\hat{x}_t$, an information state.

We need the following definition.

Definition 7.2. (Equivalent information state) A process $\{\nu_t\}$ is called an equivalent information state if and only if ν_t carries the same sufficient information I_t as the information state α_t, and is determined by a recursive formula.

For the equivalent information state, we have the following theorem.

Theorem 7.2. For a given information pattern $I_t=\{Y_0{}^t,U_0{}^t\}$, the *a posteriori* pdf $p\{x_t|\mathcal{I}_t\}$ constitutes an equivalent information to α_t, i.e. $\nu_t=(p\{x_t|\mathcal{I}_t\})$. Particularly, for Σ_L $\nu_t=(p\{x_t|\mathcal{Y}_t\})$.

Proof. Obviously, the cond. pdf $p\{x_t|\mathcal{I}_t\}$ carries all the information about the information pattern $I_t=\{Y_0^t, U_0^t\}$, and this is clearly equivalent to the statistic α_t.

For Σ_L, note that $I_t=\{Y_0^t\}$. Hence $\nu_t=(p\{x_t|\mathcal{I}_t\})=(p\{x_t|\mathcal{Y}_t\})$. (Q.E.D.)

7.5. Information State and Adaptive Control Systems

Define a system Σ_{AN} with unknown parameter $\theta\varepsilon\Theta$ by

$$(7.17)\qquad \Sigma_{AN}: \begin{cases} dx(t) = f[t,x(t),\theta]dt + C(t,\theta)u(t)dt \\ \qquad\qquad\qquad\qquad\qquad + G(t)dw(t) \\ dy(t) = h[t,x(t),\theta]dt + R(t)dv(t), \end{cases}$$

where u is assumed to be a feedback control. In the Bayesian estimation theory, one tries to obtain a recursive equation for the *a posteriori* joint pdf of x_t and θ, $p\{x_t,\theta|\mathcal{I}_t\}$, which can be written as

$$(7.18)\qquad p\{x_t,\theta|\mathcal{I}_t\} = p\{x_t|\theta,\mathcal{I}_t\}p\{\theta|\mathcal{I}_t\}.$$

For the joint pdf (7.18) we have the following theorem.

Theorem 7.3. For Σ_{AN}, suppose that there exist information states β_t and γ_t for the cond. pdf's $p\{x_t|\mathcal{I}_t\}$ given θ and $p\{\theta|\mathcal{I}_t\}$, respectively. Then $p\{x_t,\theta|\mathcal{I}_t\}$ is factored as

$$(7.19)\qquad p\{x_t,\theta|\mathcal{I}_t\} = p\{x_t|\theta,\beta_t\}p\{\theta|\gamma_t\}.$$

Proof. Step 1. Since γ_t is an information state about the unknown parameter θ, the joint pdf $p\{I_t,\theta\}$ is factored as

$$(7.20)\qquad p\{I_t,\theta\} = p\{\theta,\gamma_t\}g(I_t),\quad I_t\varepsilon\mathcal{I}_t.$$

Hence

$$(7.21)\qquad p\{\theta|I_t\} = \frac{p\{I_t,\theta\}}{\int_\Theta p\{I_t,\theta\}d\theta} = \frac{p\{\theta,\gamma_t\}g(I_t)}{\int_\Theta p\{\theta,\gamma_t\}g(I_t)d\theta} = p\{\theta|\gamma_t\}.$$

Step 2. By virtue of Theorem 7.1, the pdf $p\{x_t|\theta,I_t\}$ for (particular) given θ is written as

$$(7.22)\qquad p\{x_t|\theta,I_t\} = p\{x_t|\theta,\beta_t\}.$$

Step 3. Combining (7.21) and (7.22) with (7.18), we have (7.19). (Q.E.D.)

Consider a linear system defined by

$$(7.23)\qquad \Sigma_{AL}: \begin{cases} dx(t) = A(t,\theta)x(t)dt + C(t,\theta)u(t)dt \\ \qquad\qquad\qquad\qquad\qquad + G(t)dw(t) \\ dy(t) = H(t,\theta)x(t)dt + R(t)dv(t). \end{cases}$$

For Σ_{AL} we have

<u>Proposition 7.3.</u> For Σ_{AL} the estimate $\hat{x}(t|\theta)$ given θ and $\mathcal{Y}_t$, i.e. $\hat{x}(t|\theta)\triangleq E\{x_t|\theta,\mathcal{Y}_t\}$, is an information state for the cond. pdf $p\{x_t|\theta,\mathcal{Y}_t\}$, and the modified likelihood-ratio $\Lambda(t|\theta)$ defined below is an information state for pdf $p\{\theta|\mathcal{Y}_t\}$. That is, the information states β_t and γ_t defined in Theorem 7.3 are given respectively by $\beta_t=(\hat{x}(t|\theta))$ and $\gamma_t=(\Lambda(t|\theta))$.

Proof. By the $\mathcal{Y}_t$-measurability assumption for $u(t)$, the information pattern is $I_t=\{Y_0{}^t\}$. Note that for the particular θ, $p\{x_t|\theta,\mathcal{Y}_t\}=p\{x_t|\theta,\hat{x}(t|\theta)\}$ by Theorem 7.1 and Proposition 7.1, that is $\beta_t=(\hat{x}(t|\theta))$.

For the proof of $\gamma_t=(\Lambda(t|\theta))$, it is sufficient to express $p\{\theta|\mathcal{Y}_t\}$ by the term of $\Lambda(t|\theta)$. A similar effort to do so was done by Lainiotis in [85]. Here the result is briefly obtained. Define a process

$$(7.24)\qquad d\zeta(t) = -\frac{1}{2}x'(t)H'(t,\theta)\{R(t)R'(t)\}^{-1}H(t,\theta)x(t)dt$$
$$+ x'(t)H'(t,\theta)\{R(t)R'(t)\}^{-1}dy(t),$$
$$\zeta(0) = 0.$$

Then by the representation theorem [16;54,p.176],

$$(7.25)\qquad p\{x_t,\theta|\mathcal{Y}_t\} = \frac{E\{\exp\zeta_t|x_t,\theta,\mathcal{Y}_t\}\, p\{x_t,\theta\}}{E\{\exp\zeta_t|\mathcal{Y}_t\}}$$

and by the representation theorem for given θ,

$$(7.26)\qquad p\{x_t|\theta,\mathcal{Y}_t\} = \frac{E\{\exp\zeta_t|x_t,\theta,\mathcal{Y}_t\}\, p\{x_t|\theta\}}{E\{\exp\zeta_t|\theta,\mathcal{Y}_t\}}.$$

Use of (7.25) and (7.26) gives

$$(7.27)\qquad p\{\theta|Y_t\} = \frac{p\{x_t,\theta|Y_t\}}{p\{x_t|\theta,Y_t\}}$$

$$= \frac{E\{\exp \zeta_t|\theta,Y_t\}}{\int_\Theta E\{\exp \zeta_t|\theta,Y_t\}p\{\theta\}d\theta}\, p\{\theta\},$$

where $p\{\theta\}$ is the *a priori* pdf of θ. Here it is easily proved that $E\{\exp \zeta_t|\theta,Y_t\}$ is the likelihood-ratio function $\Lambda(t|\theta)$ for given θ (see, [128,178]) defined by

$$(7.28)\qquad \Lambda(t|\theta) = \exp\{\int_0^t \hat{x}'(s|\theta)H'(s,\theta)\{R(s)R'(s)\}^{-1}dy(s)$$

$$-\frac{1}{2}\int_0^t \|H(s,\theta)\hat{x}(s|\theta)\|^2_{\{R(s)R'(s)\}^{-1}}ds\}.$$

It can be shown that $\Lambda(t|\theta)$ satisfies the Itô stochastic differential equation,

$$(7.29)\qquad d\Lambda(t|\theta) = \Lambda(t|\theta)\hat{x}'(t|\theta)H'(t,\theta)\{R(t)R'(t)\}^{-1}dy(t)$$

$$\Lambda(0|\theta) = 1.$$

A glance at (7.27), (7.28) and (7.29) shows that $\Lambda(t|\theta)$ is an information state about θ, i.e. $\gamma_t=(\Lambda(t|\theta))$. (Q.E.D.)

Remark 7.3: The assertion, $\gamma_t=(\Lambda(t|\theta))$, in Proposition 7.3 holds also for Σ_{AN}.

Proposition 7.4. If for the system Σ_{AN} with the cost functional (7.10) there exists an information state α_t in Def.7.1, then α_t is given by a pair

$$(7.30)\qquad \alpha_t = (\beta_t, \gamma_t).$$

Hence the optimal control which minimizes (7.10) is given by

$$(7.31)\qquad u^o(t) = \psi(t,\beta_t,\gamma_t).$$

Proposition 7.5. If the system is restricted to Σ_{AL} in Proposition 7.4, then

$$(7.32)\qquad \alpha_t = (\hat{x}(t|\theta),\Lambda(t|\theta))$$

and

$$(7.33)\qquad u^o(t) = \psi(t,\hat{x}(t|\theta),\Lambda(t|\theta)).$$

Proof of Proposition 7.4. Note that

$$E\{\cdot|I_t\} = \int_{E^{(n)}}(\cdot)p\{x_t|I_t\}dx_t$$
$$= \int_{E^{(n)}}\int_{\Theta}(\cdot)p\{x_t,\theta|I_t\}d\theta dx_t.$$

Use of (7.19) in Theorem 7.3 gives

$$= \int_{E^{(n)}}\int_{\Theta}(\cdot)p\{x_t|\theta,\beta_t\}p\{\theta|\gamma_t\}d\theta dx_t.$$

Then the minimal cost functional defined by (7.13) is written as

$$(7.34)\qquad V(t,I_t) = \min_{\psi_t}\ [\int_{E^{(n)}}\int_{\Theta}[\int_t^T L(s,x_s,\psi_s)ds]$$
$$\times p\{x_t|\theta,\beta_t\}p\{\theta|\gamma_t\}d\theta dx_t],$$

which shows that V is a function of t, β_t and γ_t. Hence we know that $V(t,I_t)=V(t,\beta_t,\gamma_t)$. Therefore we have (7.30) and (7.31) by Proposition 7.2. (Q.E.D.)

Proposition 7.6. The equivalent information state is $\nu_t=(p\{x_t|\theta,I_t\}, p\{\theta|I_t\})$ for Σ_{AN}, and $\nu_t=(p\{x_t|\theta,I_t\},p\{\theta|I_t\})$ for Σ_{AL}.

Proof. By Theorem 7.2, $\nu_t=(p\{x_t|I_t\})$. Since

$$p\{x_t|I_t\} = \int_{\Theta}p\{x_t|\theta,I_t\}p\{\theta|I_t\}d\theta,$$

the pdf's $p\{x_t|\theta,I_t\}$ and $p\{\theta|I_t\}$ are sufficient for $p\{x_t|I_t\}$.
Hence

$$\nu_t = (p\{x_t|\theta,I_t\},\ p\{\theta|I_t\}).$$

(Q.E.D.)

7.6. Information State and the Signal Detection Problem

In this section let us consider a newly defined system Σ_{DN}:

$$(7.35)\quad \Sigma_{DN}: \begin{cases} dx(t) = f[t,x(t)]dt + C(t)u(t)dt \\ \qquad\qquad\qquad\qquad + G(t)dw(t) \\ dy(t) = \chi h[t,x(t)]dt + R(t)dv(t). \end{cases}$$

In (7.35), χ is an indicator variable taking its values 0 or 1, with known or assumed *a priori* probabilities p_0 and $p_1=1-p_0$. For the system Σ_{DN}, as might be expected, the optimal control problem involves making the decision of the existence of the signal in observed data; that is the signal detection procedure is required. A similar linear model to Σ_{DN} was extensively used by Lainiotis and his co-workers[179,180], and a slightly modified model was used by Sunahara and the author[181; see also Chap.4].

We have a theorem analogous to Theorem 7.3.

Theorem 7.4. Suppose that for Σ_{DN} there exist information states β_t and γ_t for $p\{x_t|\mathcal{I}_t\}$ and $p\{\chi|\mathcal{I}_t\}$ respectively. Then the joint pdf $p\{x_t,\chi|\mathcal{I}_t\}$ is factored as

$$(7.36)\qquad p\{x_t,\chi|\mathcal{I}_t\} = p\{x_t|\chi,\mathcal{I}_t\}p\{\chi|\mathcal{I}_t\}.$$

Proof. The procedure of the proof is formally the same as in Theorem 7.3.

Proposition 7.7. For Σ_{DN} the modified likelihood-ratio function $\Lambda(t|\chi)$ defined below is an information state for the cond. pdf $p\{\chi|\mathcal{I}_t\}$, i.e. $\gamma_t=(\Lambda(t|\chi))$.

Proof. This follows from that of Proposition 7.3. A similar relation to (7.27) holds:

$$(7.37)\qquad p\{\chi|\mathcal{I}_t\} = \frac{p\{x_t,\chi|\mathcal{I}_t\}}{p\{x_t|\chi,\mathcal{I}_t\}}$$

$$= \frac{E\{\exp\,\eta_t|\chi,\mathcal{I}_t\}}{\sum_{i=0}^{1} p_i E\{\exp\,\eta_t|\chi=i,\mathcal{I}_t\}}\,p\{\chi\},$$

where η_t is the process determined by

$$(7.38)\qquad d\eta_t = -\frac{1}{2}\chi h'(t,x_t)(R_tR_t')^{-1}h(t,x_t)dt$$

$$+\chi h'(t,x_t)(R_tR_t')^{-1}dy(t),\qquad \eta(0)=0.$$

In the second equality of (7.37), the relation $p\{\chi\}=p_0\delta(\chi)+p_1\delta(\chi-1)$ was used. The numerator, $E\{\exp\,\eta_t|\chi,I_t\}$, is equal to the likelihood-ratio $\Lambda(t|\chi)$ defined by

$$(7.39)\qquad \Lambda(t|\chi) = \exp\{\int_0^t\chi\hat{h}'(s,x_s|\chi)(R_sR_s')^{-1}dy(s)$$

$$-\frac{1}{2}\int_0^t\|\chi\hat{h}(s,x_s|\chi)\|^2_{(R_sR_s')^{-1}}ds\}.$$

Note that $\Lambda(t|\chi=0)=1$ for $\chi=0$ and $\Lambda(t|\chi=1)$ is the usual likelihood-ratio function appearing in the detection theory (simply, $\Lambda(t)$) given by

$$(7.40)\qquad \Lambda(t) = \exp\{\int_0^t\hat{h}_1'(s,x_s)(R_sR_s')^{-1}dy(s)$$

$$-\frac{1}{2}\int_0^t\|\hat{h}_1(s,x_s)\|^2_{(R_sR_s')^{-1}}ds\}.$$

In (7.39) and (7.40), $\hat{h}(s,x_s|\chi)\triangleq E\{h(s,x_s)|\chi,I_t\}$ and $\hat{h}_1(s,x_s)\triangleq\hat{h}(s,x_s|\chi=1)$. Hence (7.37) becomes

$$(7.41)\qquad p\{\chi|I_t\} = \frac{\Lambda(t|\chi)}{p_0+p_1\Lambda(t)}\,p\{\chi\}.$$

Therefore we see that $\Lambda(t|\chi)$ (this includes $\Lambda(t)$ as a special case of $\chi=1$) is an information state, that is $\gamma_t=(\Lambda(t|\chi))$. (Q.E.D.)

Proposition 7.8. For Σ_{DN} the conditional mean $\hat{\chi}(t)\triangleq E\{\chi|I_t\}$ constitutes an information state for $p\{\chi|I_t\}$, i.e. $\gamma_t=(\hat{\chi}(t))$.

Proof. By (7.41) the *a posteriori* probability $P[\chi=1|I_t]$ is evaluated as

$$(7.42)\qquad P[\chi=1|I_t] = \frac{p_1\Lambda(t)}{p_0+p_1\Lambda(t)}.$$

Obviously

$$(7.43)\qquad \hat{\chi}(t) = E\{\chi|I_t\} = P[\chi=1|I_t]$$

$$= \frac{\rho\Lambda(t)}{1+\rho\Lambda(t)},$$

where $\rho=p_1/p_0$. Since by Proposition 7.7 the likelihood-ratio $\Lambda(t)$ constitutes an information state for $p\{\chi|I_t\}$, thus $\hat{\chi}(t)$ also constitutes an information state. (Q.E.D.)

7.7. Summaries and Discussions

So far we have investigated the conditions and properties of sufficient statistics α_t (or β_t and γ_t) and ν_t. The role of sufficient statistics is the data reduction of information pattern I_t, which consists of $\{y(s),\ 0\le s\le t\}$ and $\{u(s),\ 0\le s\le t\}$, by the replacement of an information state. The consequences are summarized in Tables 7.1 and 7.2. Table 7.1 shows the conditions of information states for various types of stochastic systems. Some typical information states are listed in Table 7.2.

As is well-known, except for the LQG (linear-quadratic-Gaussian) problem, the information state α_t is in general unknown and hence the optimal control cannot be obtained in practice. To see this, define $m_t=E\{x_t|I_t\}$ and $m_{it}=E\{(x_t-m_t)^i|I_t\}$ where the one-dimensional case is considered. Then, the cond. pdf $p\{x_t|I_t\}$ can be represented by a function

Table 7.1. Condition for Information States.

Dynamical System	Condition for Information States	Remark
Σ_L, Σ_N	$p\{x_t|I_t\} = p\{x_t|\alpha_t\}$	Theorem 7.1
Σ_{AL}, Σ_{AN}	$p\{x_t,\theta|I_t\} = p\{x_t|\theta,\beta_t\}p\{\theta|\gamma_t\}$	Theorem 7.3
Σ_{DL}, Σ_{DN}	$p\{x_t,\chi|I_t\} = p\{x_t|\chi,\beta_t\}p\{\chi|\gamma_t\}$	Theorem 7.4

Table 7.2. Information States.

Dynamical System	Information Pattern	Information States		Remarks
Σ_L	$I_t=\{Y_0{}^t\}$	α_t	$\hat{x}(t\mid t)$	Prop.7.1
		ν_t	$p\{x_t\mid Y_t\}$	Theorem 7.2
Σ_N	$I_t=\{Y_0^t, U_0^t\}$	α_t	unknown	
		ν_t	$p\{x_t\mid I_t\}$	Theorem 7.2
Σ_{AL}	$I_t=\{Y_0{}^t\}$	β_t	$\hat{x}(t\mid\theta)$	Prop.7.3
		γ_t	$\Lambda(t\mid\theta)$	Prop.7.3
		ν_t	$(p\{x_t\mid\theta,Y_t\},\ p\{\theta\mid Y_t\})$	Prop.7.6
Σ_{AN}	$I_t=\{Y_0^t, U_0^t\}$	β_t	unknown	
		γ_t	$\Lambda(t\mid\theta)$	Prop.7.3, Remark 7.3
		ν_t	$(p\{x_t\mid\theta,I_t\},\ p\{\theta\mid I_t\})$	Prop.7.6
Σ_{DL}	$I_t=\{Y_0{}^t\}$	β_t	$\hat{x}(t\mid\chi)$	(Prop.7.3)
		γ_t	$\Lambda(t\mid\chi)$ or $\hat{\chi}(t)$	(Prop.7.7 & 7.8)
		ν_t	$(p\{x_t\mid\chi=1,Y_t\},\ p\{\chi=1\mid Y_t\})$	(Prop.7.6)
Σ_{DN}	$I_t=\{Y_0^t, U_0^t\}$	β_t	unknown	
		γ_t	$\Lambda(t\mid\chi)$ or $\hat{\chi}(t)$	(Prop.7.7 & 7.8)
		ν_t	$(p\{x_t\mid\chi=1,Y_t\},\ p\{\chi=1\mid Y_t\})$	(Prop.7.6)

The parenthesis (·) in the remark column means that the listed result easily follows from " · ".

of infinite moments $\{m_t,m_{2t},m_{3t},\cdots\}$, i.e.

(7.44) $$p\{x_t\mid I_t\} = \phi(x_t,m_t,m_{2t},m_{3t},\cdots),$$

Hence, the information state α_t will be presented by

$$\alpha_t = \alpha(t,m_t,m_{2t},m_{3t},\cdots),$$

for which further considerations stop here. Furthermore, since the form of optimal control (7.11) is given by

$$(7.45) \qquad u^{o}(t) = \psi(t, m_t, m_{2t}, m_{3t}, \cdots),$$

the precise realization of the optimal control for the nonlinear system Σ_N is impossible. However, for the stochastic control of nonlinear systems, there are some papers in which the information state α_t is approximated by $\hat{x}_t$ on an analogy of the linear case. For example, using the wide-sense property by Doob[28], Tse, Bar-Shalom and Meier[146] and Tse and Bar-Shalom[182] obtained the practical control for systems similar to Σ_N and Σ_{AN} which is referred to as a wide-sense adaptive control law. Alternatively, Sunahara[183] and Sunahara and the author [129,130] obtained the suboptimal control for Σ_N, using the concept of stochastic linearization in Markovian framework. In the papers [129,130, 183], the *a posteriori* cond. pdf $p\{x_t | I_t\}$ was approximated to be Gaussian and the information state was assumed to be $\alpha_t = (\hat{x}_t)$.

The study of information states is extremely important in the field of stochastic nonlinear control systems. There has been little study of the best approximation of the information state and of the asymptotic information state which will be useful for a long-term control.

CHAPTER 8. CONCLUSIONS

8.1. Concluding Remarks

In Part One, a feasible method of signal detection and estimation-control has been established in a form of computer-aided feedback system for a wide class of nonlinear stochastic systems under noisy observations. The basic notion of suboptimal control for nonlinear systems is use of stochastic linearization technique reviewed in Chap.3. It should be particularly emphasized that the stochastic linearization method plays a useful role to the realization of computer-oriented estimation-control system.

There are, in general, two possibilities of linearization in nonlinear systems as pointed out by Tsypkin and by Kashyap (cf.[27]) as

(a) linearization of the nonlinear element only

(b) linearization of the nonlinear system as a whole.

More concretely, the category (b) may be devided into the following three parts;

(b-i) linearization of the nonlinear system dynamics (b-ii) linearization of the filter dynamics

(b-iii) linearization of the basic Hamilton-Jacobi-Bellman equation. The technique in the category (a) mainly plays a useful role to analize and synthesize a simple system with the single input-output relation which contains a nonlinear element of zero-memory type. However, in the case of complexed large nonlinear systems, the linearization of nonlinear element requires tediously complexed computation. Therefore, from global viewpoints, the category (b) will be more preferable. By invoking the linearization technique for a nonlinear system as a whole, it is easy to obtain the approximated behavior of nonlinear systems as a birdview picture.

In particular, the linearization of a whole system may enthusiastically recommended in the case of optimal control of complexed nonlinear systems with/without noisy observation. In such the case, there are the three sub-categories stated above. Among them, it may be stated that the linearization of nonlinear dynamics, (b-i), is pleasant. Thus the stochastic linearization technique may be emphasized in constructing the overall configuration of a broad class of stochastic optimal control.

8.2. Discussions

In the theory of stochastic control, it is a primary problem to find the "informative" quantity for control. The informative quantity is an important concept of sufficient statistics, and is the summary of a large amount of such data as observations up to the present time, all the *a priori* knowledges of the system and the past control in describing the future evolution of the system.

For linear control systems, the sufficient statistics was studied for the purpose of data reduction by Striebel[125], Aoki[2], Bohlin[12], and Davis and Varaiya[25], forcing us to look deeper into its mathematical importance in the optimal control of stochastic systems.

Up to the present time, the strict optimal control of nonlinear system is still impossible. The ultimate reason is due to the "curse of dimensionality"[9] which prevents us to use the dynamic programming. Therefore, the study of sufficient statistics is extremely important in the field of stochastic nonlinear control systems. In Chap.7, the author has studied the conditions of the sufficient statistics which is called in terms of "information state." There has been yet little study about the best approximation to the sufficient statistics for nonlinear systems. Although Part One will contribute to the study of nonlinear control systems, the study of sufficient statistics and its approximation will be one of the topics of current researches.

APPENDIX A. Typical Examples of Stochastic Linearization

This appendix serves several typical examples of the stochastic linearization technique which is reviewed in Chap.3.

Example A.1. Saturation Element.

Consider the one-dimensional case. The nonlinear function $f(t,x)$ is given

$$f(t,x) = \begin{cases} A & \text{for } x>A \\ x & \text{for } |x| \leq A \\ -A & \text{for } x<-A. \end{cases} \tag{A.1}$$

From (3.3a), it follows that

$$a(t) = E\{f(t,x)|Y_t\} = \int_{-\infty}^{\infty} f(t,x)p\{t,x|Y_t\}dx, \tag{A.2}$$

where $p\{t,x|Y_t\}$ is the conditional probability density function which is assumed to be

$$p\{t,x|Y_t\} = \frac{1}{\sqrt{2\pi p(t|t)}}\exp\{-\frac{(x-\hat{x})^2}{2p(t|t)}\}. \tag{A.3}$$

For the nonlinear function (A.1), the coefficient $a(t)$ becomes

$$\begin{aligned} a(t) &= \int_{-\infty}^{-A}(-A)\frac{1}{\sqrt{2\pi p}}\exp\{-\frac{(x-\hat{x})^2}{2p}\}dx \\ &+ \int_{-A}^{A}x\frac{1}{\sqrt{2\pi p}}\exp\{-\frac{(x-\hat{x})^2}{2p}\}dx \\ &+ \int_{A}^{\infty}A\frac{1}{\sqrt{2\pi p}}\exp\{-\frac{(x-\hat{x})^2}{2p}\}dx. \end{aligned} \tag{A.4}$$

The first integral on the right-hand side of (A.4) is

$$\begin{aligned} I_1 &= \int_{-\infty}^{-A}(-A)\frac{1}{\sqrt{2\pi p}}\exp\{-\frac{(x-\hat{x})^2}{2p}\}dx = -\frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} e^{-\frac{\lambda^2}{2}}d\lambda \\ &= -A\Phi(\alpha), \end{aligned} \tag{A.5}$$

where

$$\alpha = -\frac{A+\hat{x}}{\sqrt{p}}, \tag{A.6}$$

and

(A.7) $$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{\lambda^2}{2}} d\lambda.$$

The second integral is computed to be

(A.8) $$I_2 = \int_{-A}^{A} x\frac{1}{\sqrt{2\pi p}}\exp\{-\frac{(x-\hat{x})^2}{2p}\}dx$$

$$= \frac{1}{\sqrt{2\pi p}}\int_{\alpha}^{-\beta}(\hat{x}+\sqrt{p}\lambda)e^{-\frac{\lambda^2}{2}}\sqrt{p}d\lambda$$

$$= \frac{1}{\sqrt{2\pi}}\hat{x}\int_{\alpha}^{-\beta}e^{-\frac{\lambda^2}{2}}d\lambda + \frac{1}{\sqrt{2\pi}}\sqrt{p}\int_{\alpha}^{-\beta}\lambda e^{-\frac{\lambda^2}{2}}d\lambda,$$

where

(A.9) $$\beta = -\frac{A-\hat{x}}{\sqrt{p}}.$$

A simple calculation shows that

(A.10a) $$\int_{\alpha}^{-\beta}e^{-\frac{\lambda^2}{2}}d\lambda = \sqrt{2\pi}[1-\Phi(\alpha)-\Phi(\beta)]$$

(A.10b) $$\int_{\alpha}^{-\beta}\lambda e^{-\frac{\lambda^2}{2}}d\lambda = e^{-\frac{\alpha^2}{2}} - e^{-\frac{\beta^2}{2}}.$$

Thus

(A.11) $$I_2 = \hat{x}[1-\Phi(\alpha)-\Phi(\beta)] + \sqrt{\frac{p}{2\pi}}(e^{-\frac{\alpha^2}{2}}-e^{-\frac{\beta^2}{2}}).$$

The third integral becomes

(A.12) $$I_3 = \int_{A}^{\infty}A\frac{1}{\sqrt{2\pi p}}\exp\{-\frac{(x-\hat{x})^2}{2p}\}dx = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\beta}e^{-\frac{\lambda^2}{2}}d\lambda$$

$$= A\Phi(\beta).$$

Then combining (A.5), (A.8) and (A.12), with (A.14), it follows that

(A.13) $$a(t) = \hat{x} + \alpha\sqrt{p}\Phi(\alpha) - \beta\sqrt{p}\Phi(\beta) + \sqrt{\frac{p}{2\pi}}(e^{-\frac{\alpha^2}{2}}-e^{-\frac{\beta^2}{2}})$$

$$= \frac{1}{2}\sqrt{p}\,[\alpha\,\mathrm{erf}(\tfrac{\alpha}{\sqrt{2}}) - \beta\,\mathrm{erf}(\tfrac{\beta}{\sqrt{2}})] + \sqrt{\frac{p}{2\pi}}\,(e^{-\frac{\alpha^2}{2}} - e^{-\frac{\beta^2}{2}}),$$

where the following relation has been used:

$$\text{(A.14)} \qquad \Phi(x) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}(\tfrac{x}{\sqrt{2}}).$$

The other coefficient $b(t)$ is computed by using the relation (3.6), i.e. $b(t)=\partial a(t)/\partial\hat{x}$. From (A.6), (A.9) and (A.13), the coefficients are obtained as

$$\text{(A.15)} \qquad a(t) = \frac{1}{2}[(A+\hat{x})\mathrm{erf}(\tfrac{A+\hat{x}}{\sqrt{2p}}) - (A-\hat{x})\mathrm{erf}(\tfrac{A-\hat{x}}{\sqrt{2p}})]$$

$$+ \sqrt{\frac{p}{2\pi}}[\exp\{-\frac{(A+\hat{x})^2}{2p}\} - \exp\{-\frac{(A-\hat{x})^2}{2p}\}]$$

and

$$\text{(A.16)} \qquad b(t) = \frac{1}{2}[\mathrm{erf}(\tfrac{A+\hat{x}}{\sqrt{2p}}) + \mathrm{erf}(\tfrac{A-\hat{x}}{\sqrt{2p}})].$$

Example A.2. On-Off Element.

The nonlinear function is given by

$$\text{(A.17)} \qquad f(t,x) = \begin{cases} A & \text{for } x>0 \\ -A & \text{for } x<0. \end{cases}$$

This case is similar to Example A.1. The $a(t)$ is evaluated to be

$$\text{(A.18)} \qquad a(t) = \int_{-\infty}^{0}(-A)p\{t,x|Y_t\}dx + \int_{0}^{\infty}Ap\{t,x|Y_t\}dx.$$

The first and second integrals are:

$$\text{(A.19a)} \qquad I_1 = -A\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi p}}\exp\{-\frac{(x-\hat{x})^2}{2p}\}dx = -\frac{A}{\sqrt{2}}\int_{-\infty}^{\alpha}e^{-\frac{\lambda^2}{2}}d\lambda$$

$$= A\Phi(-\alpha),$$

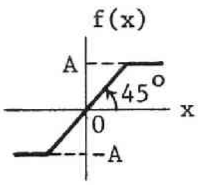
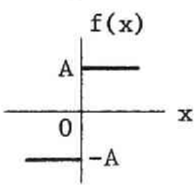
where

$$\text{(A.20)} \qquad \alpha = -\frac{\hat{x}}{\sqrt{p}}.$$

Thus

$$\text{(A.21)} \qquad a(t) = -A\Phi(\alpha) + A\Phi(-\beta)$$

Table A.1. Coefficients of Stochastic Linearization for Typical Examples (one-dimensional case)

Nonlinear Function $f(t,x)$		$a(t)$	$b(t)$
x^n	$n=2$	$\hat{x}^2+p$	$2\hat{x}$
	$n=3$	$\hat{x}^3+3\hat{x}p$	$3(\hat{x}^2+p)$
	$n=4$	$\hat{x}^4+6\hat{x}^2p+3p^2$	$4(\hat{x}^3+3\hat{x}p)$
	$n=5$	$\hat{x}^5+10\hat{x}^3p+15\hat{x}p^2$	$5(\hat{x}^4+6\hat{x}^2p+3p^2)$
	$n=6$	$\hat{x}^6+15\hat{x}^4p+45\hat{x}^2p^2+15p^3$	$6(\hat{x}^5+10\hat{x}^3p+15\hat{x}p^2)$
	$n=7$	$\hat{x}^7+21\hat{x}^5p+105\hat{x}^3p^2+105\hat{x}p^3$	$7(\hat{x}^6+15\hat{x}^4p+45\hat{x}^2p^2+15p^3)$
	...	...	...
$\sin x$		$\sin\hat{x}\exp(-\frac{p}{2})$	$\cos\hat{x}\exp(-\frac{p}{2})$
$\cos x$		$\cos\hat{x}\exp(-\frac{p}{2})$	$-\sin\hat{x}\exp(-\frac{p}{2})$
$x\sin x$		$(\hat{x}\sin\hat{x}+p\cos\hat{x})\exp(-\frac{p}{2})$	$[(1-p)\sin\hat{x}+\hat{x}\cos\hat{x}]\exp(-\frac{p}{2})$
Example 1 		$\frac{1}{2}[(A+\hat{x})\mathrm{erf}(\frac{A+\hat{x}}{\sqrt{2p}})-(A-\hat{x})\mathrm{erf}(\frac{A-\hat{x}}{\sqrt{2p}})] + \sqrt{\frac{p}{2\pi}}[\exp\{-\frac{(A+\hat{x})^2}{\sqrt{2p}}\}-\exp\{-\frac{(A-\hat{x})^2}{\sqrt{2p}}\}]$	$\frac{1}{2}[\mathrm{erf}(\frac{A+\hat{x}}{\sqrt{2p}})+\mathrm{erf}(\frac{A-\hat{x}}{\sqrt{2p}})]$
Example 2 		$A\,\mathrm{erf}(\frac{\hat{x}}{\sqrt{2p}})$	$A\sqrt{\frac{2}{\pi p}}\exp\{-\frac{\hat{x}^2}{2p}\}$

$$= -A[\frac{1}{2} - \frac{1}{2}\mathrm{erf}(-\frac{\alpha}{\sqrt{2}})] + A[\frac{1}{2} + \frac{1}{2}\mathrm{erf}(-\frac{\alpha}{\sqrt{2}})]$$

$$= A\,\mathrm{erf}(-\frac{\alpha}{\sqrt{2}}) = A\,\mathrm{erf}(\frac{\hat{x}}{\sqrt{2p}}),$$

where (A.14) has been used again. Simple calculation shows that*

$$\text{(A.22)} \qquad b(t) = A\sqrt{\frac{2}{\pi p}}\exp\{-\frac{\hat{x}^2}{2p}\}.$$

APPENDIX B. Derivation of Eq.(4.17).

Write (4.16) as

$$\text{(B.1)} \qquad \sum_{j=0}^{k-1} P(H_j)\{f_{ij}(Y_0^t)-f_{\nu j}(Y_0^t)\}p\{Y_0^t|H_j\}$$

$$\leq P(H_{-1})\{f_{\nu\,-1}(Y_0^t)-f_{i\,-1}(Y_0^t)\}p\{Y_0^t|H_{-1}\}.$$

Note that

$$\text{(B.2)} \qquad \{f_{ij}(Y_0^t)-f_{\nu j}(Y_0^t)\} = \{f_{-1j}(Y_0^t)-f_{\nu j}(Y_0^t)\} - \{f_{-1j}(Y_0^t)-f_{ij}(Y_0^t)\}$$

and

$$\text{(B.3)} \qquad \{f_{\nu\,-1}(Y_0^t)-f_{i\,-1}(Y_0^t)\} = \{f_{\nu\,-1}(Y_0^t)-f_{-1\,-1}(Y_0^t)\}$$

$$- \{f_{i\,-1}(Y_0^t)-f_{-1\,-1}(Y_0^t)\}.$$

Substituting (B.2) and (B.3) into (B.1) and rearranging terms with uses of (4.4) and (4.12), we have

$$\text{(B.4)} \qquad \sum_{j=0}^{k-1}\{f_{-1j}(Y_0^t)-f_{\nu j}(Y_0^t)\}\Lambda(t,t_j) - \rho_k\{f_{\nu\,-1}(Y_0^t)-f_{-1\,-1}(Y_0^t)\}$$

$$\leq \sum_{j=0}^{k-1}\{f_{-1j}(Y_0^t)-f_{ij}(Y_0^t)\}\Lambda(t,t_j) - \rho_k\{f_{i\,-1}(Y_0^t)-f_{-1\,-1}(Y_0^t)\}$$

---

* In the evaluation of (A.23), the following formula has been used:
$$\frac{\partial}{\partial x}\,\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\exp(-x^2).$$

from which (4.17) results.

APPENDIX C. Derivation of the Likelihood-Ratio.

The likelihood-ratio $\Lambda(t,t_j)$ defined by (4.4) is obtained by considering the following detection problem between two hypotheses

$$\text{(C.1)} \quad H_j: \quad dy(\tau) = \begin{cases} R(\tau)dv(\tau) & 0\leq\tau<t_j \\ H(\tau)x(\tau)d\tau + R(\tau)dv(\tau) & t_j\leq\tau\leq t_k \end{cases}$$

(Signal exists.)

$$\text{(C.2)} \quad H_{-1}: \quad dy(\tau) = R(\tau)dv(\tau) \quad 0\leq\tau\leq t_k$$

(Signal does not exist.).

In (C.1), the state variable $x(\tau)$ is the solution of the differential equation under H_j,

$$\text{(C.3)} \qquad dx(\tau) = A(\tau)x(\tau)d\tau + G(\tau)dw(\tau)$$

with $x(t_j)=x_0$.

Partition the interval $[0,t]$,

$$0 = s_0 < s_1 < \cdots < s_K = t,$$

such that this partition includes I, and let $\varepsilon=\max_i (s_{i+1}-s_i)$. Construct the conditional density $p\{y_{s_0},\cdots,y_{s_K}|H_j\}$ such that

$$\text{(C.4)} \qquad p\{Y_0^t|H_j\} = \underset{\substack{\varepsilon\to 0 \\ K\to\infty}}{\text{l.i.m.}}\ p\{y_{s_0},\cdots,y_{s_K}|H_j\}.$$

Then, we have for $p\{y_{s_0},\cdots,y_{s_K}|H_j\}$,

$$\text{(C.5)} \qquad p\{y_{s_0},\cdots,y_{s_K}|H_j\} = E[p\{y_{s_0},\cdots,y_{s_K}|\{x(s),s\varepsilon[t_j,t]\},H_j\}]$$

$$= c_0 \exp\{-\frac{1}{2}\sum_{\nu=0}^{j-1}\frac{1}{\delta s_\nu}\|\delta y(s_\nu)\|^2_{\{R(s_\nu)R'(s_\nu)\}^{-1}}\}$$

$$\times E[\exp\{-\frac{1}{2}\sum_{\nu=j}^{K}\frac{1}{\delta s_\nu}\|\delta y(s_\nu)-H(s_\nu)x(s_\nu)\delta s_\nu\|^2_{\{R(s_\nu)R'(s_\nu)\}^{-1}}\}],$$

where c_0 is a normalizing coefficient. Also for $p\{y_{s_0},\cdots,y_{s_K}|H_{-1}\}$, we have

$$(C.6)\qquad p\{y_{s_0},\cdots,y_{s_K}|H_{-1}\} = c_0 \exp\{-\frac{1}{2}\sum_{\nu=j}^{K}\frac{1}{\delta s_\nu}\|\delta y(s_\nu)\|^2_{\{R(s_\nu)R'(s_\nu)\}^{-1}}\}.$$

Dividing (C.5) by (C.6) and cancelling the terms, it follows that

$$(C.7)\qquad \frac{p\{y_{s_0},\cdots,y_{s_K}|H_j\}}{p\{y_{s_0},\cdots,y_{s_K}|H_{-1}\}}$$

$$= E[\exp\{\sum_{\nu=j}^{K} x'(s_\nu)H'(s_\nu)\{R(s_\nu)R'(s_\nu)\}^{-1}\delta y(s_\nu)$$

$$-\frac{1}{2}\sum_{\nu=j}^{K}\|H(s_\nu)x(s_\nu)\|^2_{\{R(s_\nu)R'(s_\nu)\}^{-1}}\delta s_\nu\}]$$

$$= \exp\{\sum_{\nu=j}^{K}\hat{x}_j'(s_\nu|s_\nu)H'(s_\nu)\{R(s_\nu)R'(s_\nu)\}^{-1}\delta y(s_\nu)$$

$$-\frac{1}{2}\sum_{\nu=j}^{K}\|H(s_\nu)\hat{x}_j'(s_\nu|s_\nu)\|^2_{\{R(s_\nu)R'(s_\nu)\}^{-1}}\delta s_\nu\}.$$

From (C.7) we have

$$(C.8)\qquad \Lambda(t,t_j) = \underset{\substack{\varepsilon\to 0\\ K\to\infty}}{\text{l.i.m.}}\ \exp\{\sum_{\nu=j}^{K}\hat{x}_j'(s_\nu|s_\nu)H'(s_\nu)\{R(s_\nu)R'(s_\nu)\}^{-1}\delta y(s_\nu)$$

$$-\frac{1}{2}\sum_{\nu=j}^{K}\|H(s_\nu)\hat{x}_j(s_\nu|s_\nu)\|^2_{\{R(s_\nu)R'(s_\nu)\}^{-1}}\delta s_\nu\}$$

$$= \exp\{\int_{t_j}^{t}\hat{x}_j'(s|s)H'(s)\{R(s)R'(s)\}^{-1}dy(s)$$

$$-\frac{1}{2}\int_{t_j}^{t}\|H(s)\hat{x}_j(s|s)\|^2_{\{R(s)R'(s)\}^{-1}}ds\}.$$

This completes the proof. (Q.E.D.)

APPENDIX D. Cost Assignments.

Let us define the following set of quadratic cost functions D in (4.5),

for $i=j=-1$ $\quad D[x(s),0,H_{-1}] = 0$

$$i=-1,\ j\neq-1 \qquad D[x(s),0,H_j] = c_1\|x_j(s)\|^2$$

$$i\neq-1,\ j=-1 \qquad D[x(s),\hat{x}_i(s|s),H_{-1}] = c_2\|x(s)-\hat{x}_i(s|s)\|^2$$

$$i\neq-1,\ j\neq-1 \qquad D[x(s),\hat{x}_i(s|s),H_j] = c_3\|x_j(s)-\hat{x}_i(s|s)\|^2,$$

where $x_j(s)$ is the solution process of (4.3) with its initial time $\tau_0=t_j$; and c_1, c_2, c_3 are weighting constants. Define also the time interval S_{ij} as: $S_{-1\,-1}=[t_j,t]$, $S_{i\,-1}=[t,t+T_1]$ and $S_{ij}=[t,t+T_1]$, where T_1 is constant. Then, by (4.6) $f_{ij}(Y_0^t)$ are respectively as follows:

$$\text{(D.1)} \qquad f_{-1\,-1}(Y_0^t) = 0$$

$$\text{(D.2)} \qquad f_{-1j}(Y_0^t) = \int_{t_j}^{t} E\{c_1\|x_j(s)\|^2|Y_0^t,H_j\}ds$$

$$\cong c_1(t-t_j)E\{\|x_j(t)\|^2|Y_0^t,H_j\} = c_1(t-t_j)[\|\hat{x}_j(t|t)\|^2+\text{tr.}P_j(t|t)]$$

$$\text{(D.3)} \qquad f_{ij}(Y_0^t) = \int_{t}^{t+T_1} E\{c_3\|x_j(\tau)-\hat{x}_i(\tau|\tau)\|^2|Y_0^t,H_j\}d\tau$$

$$\cong c_3T_1E\{\|x_j(t)-\hat{x}_i(t|t)\|^2|Y_0^t,H_j\} = c_3T_1\text{tr.}Q_{ij}(t|t),$$

where $Q_{ij}(t|t)\triangleq E\{[x_j(t)-\hat{x}_i(t|t)][x_j(t)-\hat{x}_i(t|t)]'|Y_0^t,H_j\}$ and this is obtained by

$$\text{(D.4)} \qquad Q_{ij}(t|t) = P_j(t|t) + [\hat{x}_j(t|t)-\hat{x}_i(t|t)][\hat{x}_j(t|t)-\hat{x}_i(t|t)]'.$$

$$f_{i\,-1}(Y_0^t) = \int_{t}^{t+T_1} E\{c_2\|x(\tau)-\hat{x}_i(\tau|\tau)\|^2|Y_0^t,H_{-1}\}d\tau$$

$$= c_2\int_{t}^{t+T_1}\sum_{\nu=k}^{N-1} P(H_\nu|H_{-1})E\{\|x(\tau)-\hat{x}_i(\tau|\tau)\|^2|Y_0^t,H_\nu\}d\tau$$

$$= \frac{c_2}{N-k}\sum_{\nu=k}^{N-1} E\{\int_{t}^{t+T_1}\|x(\tau)-\hat{x}_i(\tau|\tau)\|^2 d\tau|Y_0^t,H_\nu\},$$

where the relations

$$E\{\cdot|Y_0^t,H_{-1}\} = \sum_{\nu=k}^{N-1} E\{\cdot|Y_0^t,H_\nu\}P(H_\nu|H_{-1})$$

and

$$P(H_\nu | H_{-1}) = P(H_\nu)/P(H_{-1}) = 1/\rho_k = 1/(N-k)$$

are used. Furthermore

$$\text{(D.5)} \qquad f_{i\,-1}(Y_0^t) = \frac{c_2}{N-k} \sum_{\nu=k}^{N-1} [E\{\int_t^{t_\nu} \|\hat{x}_i(\tau|\tau)\|^2 d\tau | Y_0^t, H_\nu\}$$

$$+ E\{\int_{t_\nu}^{t+T_1} \|x_\nu(\tau) - \hat{x}_i(\tau|\tau)\|^2 d\tau | Y_0^t, H_\nu\}]$$

$$\cong \frac{c_2}{N-k} \sum_{\nu=k}^{N-1} [(t_\nu - t)\|\hat{x}_i(t|t)\|^2$$

$$+ (t+T_1-t_\nu) E\{\|x_\nu(t_\nu) - \hat{x}_i(t|t)\|^2 | Y_0^t, H_\nu\}]$$

$$= \frac{c_2}{N-k}\{\|\hat{x}_i(t|t)\|^2 \sum_{\nu=k}^{N-1}(t_\nu - t)$$

$$+ [p_0 + \|\hat{x}_0 - \hat{x}_i(t|t)\|^2] \sum_{\nu=k}^{N-1}(t+T_1-t_\nu)\}$$

$$\cong \frac{c_2}{2}\{(T-t)\|\hat{x}_i(t|t)\|^2 + (2T_1 - T + t)[p_0 + \|\hat{x}_0 - \hat{x}_i(t|t)\|^2]\}.$$

In the above assignments, the approximations are made from the practical point of view.

APPENDIX E. Derivation of Feedback Gains.

By the assumption (6.24),

$$\text{(E.1)} \qquad \frac{\partial V(t,\kappa)}{\partial t} = \kappa' \dot{\Pi}(t)\kappa + 2\kappa'\dot{\alpha}(t) + \dot{\beta}(t)$$

$$\text{(E.2)} \qquad \frac{\partial V(t,\kappa)}{\partial t} = 2[\Pi(t)\kappa + \alpha(t)]$$

and

$$\text{(E.3)} \qquad \frac{\partial^2 V(t,\kappa)}{\partial \kappa^2} = 2\Pi(t).$$

Substituting (E.1)-(E.3) into the stochastic Hamilton-Jacobi-Bellman equation (6.23), we have

(E.4) $$- [\kappa'\dot{\Pi}(t)\kappa+2\kappa'\dot{\alpha}(t)+\dot{\beta}(t)]$$
$$= \mathrm{tr.}\{M(t)P(t|t)\} + \kappa'M(t)\kappa + 2a'(t)[\Pi(t)\kappa+\alpha(t)]$$
$$- \frac{1}{\lambda}[\kappa'\Pi(t)+\alpha'(t)]C(t)N^{-1}(t)C'(t)[\Pi(t)\kappa+\alpha(t)]$$
$$+ \mathrm{tr.}\{\Sigma'(t)\Pi(t)\Sigma(t)\}.$$

Rearranging terms in (E.4), it follows that

(E.5) $$\kappa'[\dot{\Pi}-\frac{1}{\lambda}\Pi CN^{-1}C'\Pi+M]\kappa + 2\kappa'[\dot{\alpha}-\frac{1}{\lambda}\Pi CN^{-1}C'\alpha+\Pi a]$$
$$+ [\dot{\beta}-\frac{1}{\lambda}\alpha'CN^{-1}C'\alpha+2\alpha'a+\mathrm{tr.}\{MP\}+\mathrm{tr.}\{\Sigma'\Pi\Sigma\}] = 0.$$

In order to hold (E.5) for every κ, it is necessary to hold that

(E.6) $$\dot{\Pi} - \frac{1}{\lambda}\Pi CN^{-1}C'\Pi + M = 0$$

(E.7) $$\dot{\alpha} - \frac{1}{\lambda}\Pi CN^{-1}C'\alpha + \Pi a = 0$$

(E.8) $$\dot{\beta} - \frac{1}{\lambda}\alpha'CN^{-1}C'\alpha + 2\alpha'a + \mathrm{tr.}\{MP\} + \mathrm{tr.}\{\Sigma'\Pi\Sigma\} = 0$$

which are equations (6.28), (6.29) and (6.30).

APPENDIX F. Simulation of the Brownian Motion Process.

In this appendix, the author considers only the scalar case. The Brownian motion process $w(t)$ ($t_0 \leq t < \infty$) is related to a Gaussian white noise process $\gamma(t)$ (with zero mean) by the following well-known relation, [163,127]

(F.1a) $$dw(t) = \gamma(t)dt$$

or precisely

(F.1b) $$w(t) = \int^t \gamma(s)ds,$$

where the $w(t)$-process has the properties

(F.2) $$E\{dw(t)\} = 0 \quad \text{and} \quad E\{[dw(t)]^2\} = \sigma dt.$$

In the followings, let the parameter σ be equal to one without loss of generality.

In the digital simulation, the time interval $[t_0,\infty)$ is partitioned as

$$t_0 < t_1 < t_2 <\cdots< t_j <\cdots,$$

so that, at discrete time t_j, $\delta t_j=t_{j+1}-t_j$ $(j=0,1,2,\cdots)$ may be sufficiently short. With discretized arguments, it follows from (F.2) and (F.1a) that

$$\text{(F.3)}\qquad \delta t_j = E\{(\delta w_j)^2\} = E\{\gamma^2(t_j)\}(\delta t_j)^2,$$

where

$$\text{(F.4)}\qquad \delta w_j \overset{\sim}{=} w(t_{j+1}) - w(t_j).$$

Thus, we have

$$\text{(F.5)}\qquad E\{\gamma^2(t_j)\} = (\delta t_j)^{-1},$$

which means that the variance of $\gamma(t_j)$ is equal to $(\delta t_j)^{-1}$. If the partition of the time interval is constant, say, $\delta t_j=\Delta$(const.), then we may express the above relation as

$$\text{(F.6)}\qquad \gamma\colon N[0,\tfrac{1}{\Delta}].$$

Let us introduce a standard normal random sequence $n(t)$ which can be generated by a suitable subroutine on a digital computer, and find that the relation between n and γ which is the desired noise, that is between

$$\text{(F.7)}\qquad n\colon N[0,\,1] \quad\text{and}\quad \gamma\colon N[0,\tfrac{1}{\Delta}].$$

The variance of the n-process is evaluated by

$$\text{(F.8)}\qquad \text{variance of } n = E\{n^2\}$$

$$= \int_{-\infty}^{\infty} n^2 \frac{1}{\sqrt{2\pi\cdot 1^2}} \exp\{-\frac{n^2}{2\cdot 1^2}\}dn \equiv 1.$$

On the other hand, for the γ-process, since

$$\text{(F.9)}\qquad \int_{-\infty}^{\infty}\gamma^2\frac{1}{\sqrt{2\pi\frac{1}{\Delta}}}\exp\{-\frac{\gamma^2}{2\frac{1}{\Delta}}\}d\gamma = \frac{1}{\Delta},$$

we have

(F.10) $$1 = \Delta\frac{1}{\Delta} = \Delta\int_{-\infty}^{\infty}\gamma^2\frac{1}{\sqrt{\frac{2\pi}{\Delta}}}\exp\{-\frac{\gamma^2}{\frac{2}{\Delta}}\}d\gamma$$

$$= \int_{-\infty}^{\infty}(\sqrt{\Delta}\gamma)^2\frac{1}{\sqrt{2\pi\cdot 1^2}}\exp\{-\frac{(\sqrt{\Delta}\gamma)^2}{2\cdot 1^2}\}\sqrt{\Delta}d\gamma.$$

Comparison of (F.8) and (F.10) reveals us that the evident relation,

(F.11) $$\gamma = \frac{1}{\sqrt{\Delta}}n.$$

$$\begin{array}{ccc} \text{n-process} & & \gamma\text{-process} \\ N[0,1] & \xrightarrow[\substack{\text{Transformation} \\ \gamma = \frac{1}{\sqrt{\Delta}}n}]{} & N[0,\frac{1}{\Delta}] \end{array}$$

Thus the increment of the Brownian motion process is simulated by the following relation:

(F.12) $$\delta w = \gamma\Delta = n\sqrt{\Delta}.$$

II. PART TWO. APPROXIMATE METHODS OF STATE ESTIMATION, PARAMETER IDENTIFICATION AND CONTROL FOR NONLINEAR DISTRIBUTED PARAMETER SYSTEMS

CHAPTER 1. INTRODUCTION

Although the recent interests in control theory have concentrated mainly on systems whose dynamic behaviors are described by ordinary differential equations, less attention has been payed to the distributed parameter systems (D.P.S.). Many physical systems are intrinsically distributed, and moreover requirements of treating more complex control objects are made in view of the present trend of rapidly advancing science and technology. The dynamic behavior of systems is governed by partial differential equations, integral equations or integro-differential equations.

For randomly-excited D.P.S. described by partial differential equation, several authors have examined the problems of estimation of system states including unknown parameters and of control as a first contribution to the control theory of stochastic D.P.S. Such works are surveyed in the following subsections.

The part two will be divided into three parts: the first is the filtering problem, the second the parameter identification problem, and the third the problem of optimal control for linear and nonlinear stochastic D.P.S.

The part two is to provide two important phases: first to provide mathematical developments for filtering theory, parameter identification theory and control theory, and secondly to provide approximate method of computational implementations.

1.1. Historical Background

The historical background is divided into the following three parts.

1.1.A. Filtering Problem

There is a large number of stochastic processes whose sample paths are determined by partial differential equations for which the solution of the problem of state estimation under noisy observations is extremely important. Physical examples of such estimation problems are found in the estimation of temperature profiles in a catalytic reactor or a furnace, in the estimation of effects of random disturbance on a transmission line, the estimation of diffusions due to random excitation in environmental systems, etc.

Many studies have appeared on filtering for linear partial differential equations: Falb[35], Balakrishnan and Lions[5], Tzafestas and Nightingale [149], Thau[145], Kushner[82], and Medich[95,96]. Most of these works relied on extensions of lumped parameter ideas, and derived the filtering equation of Kalman-Bucy type described by partial integro-differential equations. A problem of similar nature was considered by Saridis and Badavas[106] who used the stochastic approximation technique.

Several trials have recently been made on the derivation of filter dynamics for nonlinear D.P.S., including proposals on a variety of approximate filter dynamics for the purpose of physical realizations by Seinfeld[112], Tzafestas and Nightingale[148,151], Seinfeld et al.[113], Hwang et al.[48], Lamont and Kumar[86], and Sunahara and the author[138]. Seinfeld[112] derived the Hamilton-Jacobi equation, based on the least square criterion, and then solved approximately by using a linearization method. Tzafestas and Nightingale[151] adopted the maximum-likelihood approach and derived an approximate filter dynamics by using the differential dynamic programming technique. Seinfeld et al.[113] showed a nonlinear filter dynamics by converting the D.P.S. into a set of lumped parameter systems with the application of a finite difference approximation and by performing a limiting operation on the spatial increment, and Hwang et al.[48] converted the estimation problem into an optimal control problem.

Expanding the results of Detchmendy and Sridhar[26], and using an invariant embedding technique by Bellman et al.[10], Lamont and Kumar[86] obtained an estimation algorithm. Introducing the Girsanov's theorem of the transformation of absolutely continuous measures to the filtering theory, Sunahara and the author[138] derived the exact filtering equation from the viewpoint of the conditional expectation, and presented a feasible method of approximation to the exact filter equation.

1.1.B. Parameter Identification

It should be noted that most of physical processes exhibit a randomness over rather broad scales of time and space. In particular, the case of parameter uncertainties frequently appears in practice, where unknown parameters are surely constant or may be supposed to be constants over the operating range.

Recently, the problems actually encountered in the parameter identification for distributed systems involve an important subject in the detection of pollution sources of environmental systems modeld by linear or nonlinear partial differential equations. In most previous schemes, identification was performed by the coupled algorithm with the state estimate. Such schemes give raise to a nonlinear filtering problem for which an approximate solution may be found by using one of approaches stated in the previous subsection 1.1.A.

Recently, some trials have been made on the identification of unknown parameters which appear in the mathematical model of D.P.S. Using integration by parts along with measurement data, a set of algebraic equations in the parameters were derived by Perdreauville and Goodson[100]. In studies by Collins and Khatri[22], Zhivoglyadov and Kaipov[166], Carpenter et al.[20] and Polis et al.[101], several different methods of finite difference, stochastic approximations and Galerkin's criterion were used respectively to yield to parameter estimates. Sunahara and the author[139] presented a new method of parameter identification by invoking the Bayesian theoretic approach. Chen and Seinfeld[168] considered the identification problem of spatially varying unknown parameters by applying the nonlinear filtering theory.

Naturally, the filtering theory of linear and nonlinear D.P.S. is the background knowledge of parameter estimate.

1.1.C. Control Problem

For the control problem of D.P.S., comprehensive and excellent surveys were published by many investigators. The first important survey effort was that of Wang[153] in 1964. Butkovsky et al.[19] presented a survey of Soviet works in the field, and separately Brogan [14] published a more comprehensive survey which included a substantial amount of tutorial material. Recently, a short, but notable survey was presented by Robinson[105] in 1971, covering a list of current papers over 250 entries. Special mention should be made on the excellent work by Lions[87] who discusses the optimal problem from the viewpoint of a pure mathematician.

In the following, reviewing recent works on the optimal control problems, descriptions are mainly restricted to stochastic and/or nonlinear problems.

Significant advances in stochastic control were made by Kolb[72], Kushner[81], Tzafestas and Nightingale[150], and Sholar[117]. Most of all these works consider optimal control problems with use of extensions of lumped parameter ideas and have involved only linear systems, as might be expected. Kushner[81] showed that for a random parabolic systems with control which is a linear function of the state, the optimal regulator is determined by a Riccatian equation, based on a mathematical models described by an Itô differential equation. Also, Tzafestas and Nightingale showed that the result is a pair of linear optimal feedback controllers, their common weighting function being described by a matrix partial integro-differential equation of the Riccatian form. When the system state is not exactly measured, Sholar[117] showed that the distributed Kalman filter is imposed and that the decoupling of the optimal controllers and the optimal estimator is proved.

It is well-known that dynamical systems to be controlled exhibit sometimes nonlinear characteristics. In recent years, the optimal control of such a nonlinear D.P.S. has received considerable attentions. However, a paucity of works on stochastic systems and a lack of consideration of nonlinear problems prevent us to study the problem of optimal stochastic control for nonlinear D.P.S. An important contribution was given by Egorov[32,33] to obtain a formulation of the necessary conditions for optimality being equivalent with the formulation of Pontryagin's Maximum Principle[102]. Golub'[46] considered also the optimal control of systems described by nonlinear partial differential equations and proposed an algorithm for approximate calculation of optimal control. In [87], Lions delt with some problems in which the systems are nonlinear with respect to controls, and derived necessary conditions on the optimal controls. Yavin[164] derived sufficient conditions for two classes of nonlinear D.P.S.; and Fjeld and Kristiansen[37] obtained conditions for local optimality, using simple calculations of variations, and considered the optimization of a periodic process which consists of a tubular reactor. Tzafestas[147] treated the optimal final-value control problem for fully nonlinear composite distributed- and lumped-parameter systems, and obtained an iterative computational algorithm. Expanding the stochastic linearization technique to D.P.S., Sunahara and the author [141,142] made an effort to obtain a suboptimal control for a general class of nonlinear D.P.S. subjected by disturbances, and explored computational algorithm for implementing the results.

1.2. Problem Considered

In Part Two, we consider the problems of estimation of system state, parameter identification and/or optimal control for a general class of nonlinear distributed parameter systems subjected to disturbances, and develop the implementation technique for the results. Physical systems under consideration are shown in Fig.1.1. Environmental effects on the system are represented by a set of disturbances (noises). The observation mechanism corresponds to a set of transducers or measuring instruments which monitor system states and transform them into a set of output quantities.

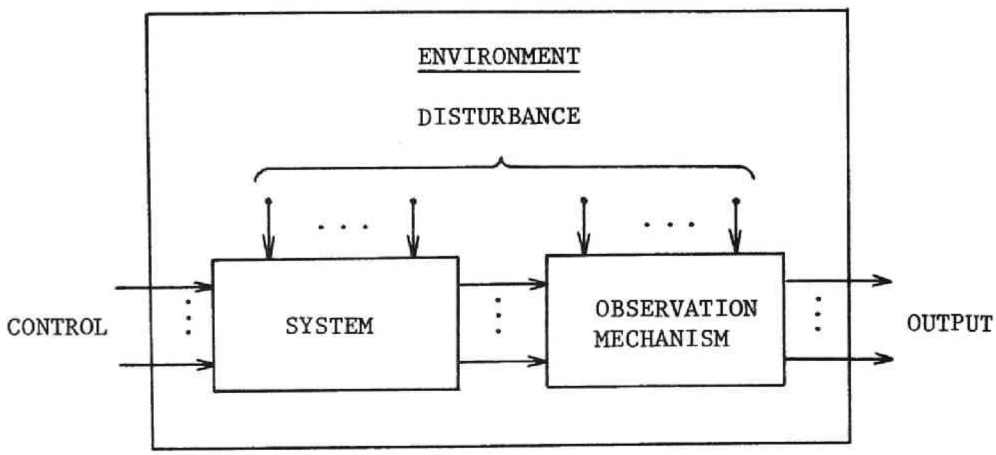


Fig.1.1. System description.

The dynamic behavior of a large number of D.P.S. can be described by a partial differential equation of the form:

$$(1.1)\qquad \frac{\partial u(t,x)}{\partial t} = F(t,x,u,\partial u/\partial x,\partial^2 u/\partial x^2;\theta) + C(t,x)f(t,x) + G(t,x,u)\gamma(t,x), \quad x\varepsilon D$$

defined on a fixed spatial domain D for $t\varepsilon[0,T]$, where $u(t,x)$ is a scalar system state; F is a nonlinear operator; C and G are known functions; γ is a formal Gaussian white noise which represents the environmental disturvance; f is a control function; and θ in the operator F is specified as a known or unknown parameter. The system state $u(t,x)$ is observed by observation mechanism given by

$$(1.2)\qquad z(t) = \int_D H(t,x,u)dx + R(t)\zeta(t).$$

The output $z(t)$ is scalar; H is a nonlinear function; R is a parameter coefficient; and ζ is a Gaussian white noise with unit variance.

Since the models (1.1) and (1.2) are purely formal because of the existence of white noises, they are well-modeled respectively by a kind of stochastic differential equation of Itô-type;

$$(1.3)\qquad du(t,x) = F(t,x,u,\partial u/\partial x,\partial^2u/\partial x^2;\theta)dt + C(t,x)f(t,x)dt + G(t,x,u)dw(t,x)$$

and

$$(1.4)\qquad dy(t) = [\int_D H(t,x,u)dx]dt + R(t)dv(t),$$

where $y(t)$ is a scalar observation process according to the similar relation to (1.5) in Sec.1.2, Chap.1 of Part One, $z(t)=\dot{y}(t)$.

Based on the models (1.3) and (1.4), in Part Two we consider the following three intrinstic problems; i.e.

(i) Estimation of the system state $u(t,x)$ of the system (1.3) from the observation data $\{y(s),\ 0\leq s\leq t\}$ obtained by the process (1.4), in which the parameter θ is assumed to be known;

(ii) Identification of the unknown parameter θ in (1.3), which is very important in the field of the system identification;

and

(iii) Optimal control of the system (1.3).

1.3. Summary of Contents

The orientation of Part Two is first to propose the possible solution for the basic and intrinstic problems, that is the problems of estimation of the system state, parameter estimation, and optimal control, which should necessarily be considered in constructing the physical distributed parameter control system, and then to provide the proposed approximate method.

The outline of Part Two is as follows.

In Chap.2, the precise mathematical models for both the dynamical system and the observation mechanism are established.

Chapter 3 provides two possible methods of expansions of a nonlinear function. One is based on the Taylor series and the other the stochastic linearization. These methods of expansion are extensively used in Chap.4 and Chap.6.

In Chap.4, the nonlinear filtering theory is developed based on the measure-theoretic approach for a general class of nonlinear D.P.S. with a Gaussian white noise disturbance under noisy observations. The principal method is to introduce the Girsanov's measure-transformation theorem to the filtering theory. Using the differential generator extended to the case of stochastic differential equations, a version of conditional expectation is derived in a form of integro-differential equations. Also a contribution is made to the method of physical realization of nonlinear filters.

Chapter 5 contains the development of parameter identification for the purpose of detecting pollution sources of environmental systems. Unknown parameters are contained in exciting terms of system dynamics. Through the Bayesian approach, a coupled scheme of state estimation and parameter identification is proposed in Markovian framework, and demonstrated by digital simulation studies.

In Chap.6, an extensive method is presented for the control of nonlinear D.P.S. under a quadratic criterion functional. Based on the study described in Part One, the extended stochastic linearization technique to D.P.S. is used to realize the optimal control system. The feasibility of approximate method is also emphasized by a simulation experiment.

CHAPTER 2. MATHEMATICAL PRELIMINARIES

2.1. Distributed Brownian Motion Process

When one wants to describe a mathematical model for the given D.P.S. subjected to additive Gaussian white noise disturbance, it is first required how one should represent mathematically the white noise disturbance which is spatially distributed. Secondly, it is also required to make clear the relation between the spatially distributed white noise and its associated Brownian motion process.

From physical viewpoints, a spatially distributed white noise, $\gamma(t,x)$ (where x is a spatial point in a fixed domain D), is considered to possess the following two properties:

(i) For each fixed $x \in D$, the process $\gamma(t,\cdot)$ is a white noise,

(ii) for each fixed t, $\gamma(\cdot,x_1)$ and $\gamma(\cdot,x_2)$ are mutually independent random processes if $x_1 \neq x_2$ and $x_1, x_2 \in D$.

The property (ii) states that for each fixed t the function $\gamma(\cdot,x)$ has the nature of "whiteness" with respect to the spatial point. Thus since it will be clear from (i) and (ii) that the spatially distributed process $\gamma(t,x)$ has an extreme irregularity with respect to both t and x, it is almost impossible to treat it as a "usual" function of t and x. The rigorous treatment of such the process $\gamma(t,x)$ requires the Schwartz distribution theory[110] and the theory of generalized random field (cf. Gel'fant and Vilenkin[43]). Such a treatment is discussed in [144].

However, the distributed Gaussian white noise $\gamma(t,x)$ is related here to a spatially distributed Brownian motion process with an analogy of Eq.(2.2) in Sec.2.1, Chap.2 in Part One, as

$$(2.1) \qquad w(t,x) = \int^t \gamma(s,x)ds, \quad x\varepsilon D$$

where $w(t,x)$ is a distributed Brownian motion process. Clearly, $w(t,x)$ defined by (2.1) has the properties of Brownian motion process for each fixed $x\varepsilon D$. In what follows, the covariance of w is assumed to be

$$(2.2) \qquad E\{dw(t,x)dw(t,z)\} = Q(x,z)dt,$$

where $Q(x,z)$ is a nonnegative and symmetric (in x and z) function for all $x,z\varepsilon D$. If the function $Q(x,z)$ is given by

$$(2.3) \qquad Q(x,z) = Q_0\delta(x-z),$$

where Q_0 is a nonnegative constant and δ is the Dirac delta function, the Brownian motion process $w(t,x)$ is spatially independent. It may be stated that the process having the property (2.2) is milder than the property stated in (ii).

In the following discussions, we use the model (2.1) with (2.2) as the spatially distributed Brownian motion process.

2.2. System Dynamics

Let D be a bounded, open, Borel measurable, simply connected set on $E^{(n)}$, an n-dimensional Euclidean space, with closure $\bar{D}$, and ∂D be the boundary of D which is continuous and piecewise differentiable. We shall write $R=[0,T]\times D$ where $[0,T]$ is the time interval. The symbol x is an n-dimensional coordinate vector.

We shall consider a well-modeled nonlinear distributed parameter system described by

$$(2.4) \qquad u_t = F(t,x,u,u_x,u_{xx}) + G(t,x,u)\gamma(t,x)$$

with the initial condition

$$(2.5) \qquad u(0,x) = \phi(x),$$

where $u(t,x)\varepsilon R$ is a scalar function, F is a nonlinear operator, G is a known function, $\gamma(t,x)$ is a formal Gaussian white noise, and u_t, u_x and u_{xx} are partial derivatives. The version of (2.4) is interpreted more adequately by the following stochastic nonlinear partial differential equation which may be considered as an extension of the stochastic ordinary differential equation of Itô-type,

$$(2.6) \qquad du(t,x) = F(t,x,u,u_x,u_{xx})dt + G(t,x,u)dw(t,x),$$

where $w(t,x)$ is a Brownian motion process in $L^2(D)$ with the zero mean and covariance,

$$(2.7) \qquad E\{dw(t,x)dw(t,z)\} = Q(x,z)dt,$$

where the symbol $E\{\cdot\}$ denotes a mathematical expectation and $Q(x,z)$ is a symmetric nonnegative function for all $x,z\varepsilon D$.

For the purpose of mathematical security, the following assumptions are made.

Suppose that, for every $t\varepsilon[0,T]$ and $x\varepsilon D$, a σ-algebra S_t is defined, where $S_s \subset S_t$ $(s<t)$ and that a Brownian motion process is defined on R. For (2.6), the following conditions hold:

(C2.1) $F(\cdot,\cdot,\cdot,\cdot,\cdot)$ and $G(\cdot,\cdot,\cdot)$ are S_t-measurable for the fixed t and x.

(C2.2) u, u_x and u_{xx} are Hölder continuous on R.

(C2.3)* For all $t\varepsilon[0,T]$, u tends uniformly to zero as $x\to\partial D$. Furthermore, both $F(\cdot,\cdot,\cdot,\cdot,\cdot)$ and $G(\cdot,\cdot,\cdot)$ also tend uniformly to zero as $x\varepsilon\partial D$.

(C2.4) The initial value $u(0,x)$ has a bounded variance and Hölder continuous second derivatives. The initial value $u(0,x)$ is independent of $w(t,x)$.

---

* (C2.3) is for convenience of theoretical development. The problem with nonhomogeneous boundary conditions may be transformed into one with homogeneous conditions[169].

(C2.5) For all $t\varepsilon[0,T]$ and $x\varepsilon D$,

$$\int_0^T |G(t,x,\cdot)|^2 dt < \infty.$$

In the sequel, assuming that the systems (2.5)-(2.6) with its initial and boundary conditions are *bien posé* in the sense of Hadamard and that the existence of the solution is always guaranteed, we shall start with (2.6).

2.3. Observation Mechanism

Let $v(t)$ be a normalized Brownian motion process independent of the $u(t,x)$- and the $w(t,x)$-processes. The observation process $y(t)$ is the scalar random process determined by

(2.8a) $$dy(t) = [\int_D H(t,z,u(t,z))dz]dt + R(t)dv(t),$$

(2.8b) $$y(0) = 0,$$

where H is a nonlinear function with respect to $u(t,z)$ and $R(t)$ is a continuous, positive coefficient on $[0,T]$. Define

(2.9) $$h_t \triangleq \int_D H(t,z,u(t,z))dz.$$

For (2.8), the following conditions are assumed:

(C2.6) h_t is S_t-measurable for the fixed t and bounded on $[0,T]$, and

$$\int_0^T |h_t| dt < \infty, \quad \int_0^T |R(t)|^2 dt < \infty.$$

Remark 2.1: The operator $\int_D H(t,z,\cdot)dz$ is a convenient representation for scanning-type or spatial averaging-type observers[153]. If the function H is linear, i.e.

$$\int_D H(t,z,\cdot)dz = \int_D H(t,z)(\cdot)dz,$$

and further if $H(t,z)$ is replaced by $\delta(z-\eta)$ (Dirac delta function), then (2.8) shows the point-wise observation at a measuring point η. Such a case will be used in an example in Sec.5.5, Chap.5.

2.4. System Models

For convenience of the following discussions, several types of the system models which are used in Part Two of this dissertation are defined.

Definition 2.1. (System Σ_1) Let $u(t,x)$ and $y(t)$ be scalar processes of dynamical system and observation represented by

(2.10) $$du(t,x) = F(t,x,u,u_x,u_{xx})dt + G(t,x,u)dw(t,x),$$

$$\text{I.C.} \quad u(0,x) = \phi(x), \quad x\varepsilon D$$

$$\text{B.C.} \quad u(t,x) = 0, \quad x\varepsilon\partial D,$$

(2.11) $$dy(t) = [\int_D H(t,z,u)dz]dt + R(t)dv(t),$$

$$y(0) = 0,$$

where the assumptions (C2.1)-(C2.6) in Sec.2.2 and Sec.2.3 are made. Equations (2.10) and (2.11) are collectively specified as Σ_1.

Definition 2.2. (System Σ_2) Let $u(t,x)$ and $y(t)$ be scalar stochastic processes represented by

(2.12) $$du(t,x) = L_x u(t,x)dt + g(t,x,\theta)dt + G(t,x)dw(t,x),$$

$$\text{I.C.} \quad u(0,x) = \phi(x), \quad x\varepsilon D$$

$$\text{B.C.} \quad u(t,x) = 0, \quad x\varepsilon\partial D,$$

(2.13) $$dy(t) = [\int_D H(t,z)u(t,z)dz]dt + R(t)dv(t),$$

$$y(0) = 0,$$

where L_x is an elliptic operator, g is known function, and θ is a vector of unknown time-invariant parameters which is considered to be a random variable. For (2.12) and (2.13), the assumption (C2.4) in Sec.2.2 is made and

(C2.7) The coefficients of L_x and their first and second derivatives are continuous in $\bar{R}$.

(C2.8) g is bounded and continuous on [0,T].

(C2.9) For all $t\varepsilon[0,T]$, u tends uniformly to zero as $x\to\partial D$. Furthermore, both $L_x u$ and G also tend uniformly to zero as $x\to\partial D$.

(C2.10) For $x\varepsilon D$,

$$\int_0^T |G(t,x)|^2 dt < \infty.$$

Equations (2.12) and (2.13) with (C2.7)-(C2.10) are specified as Σ_2.

Futhermore the following system Σ_3 is defined.

Definition 2.3. (System Σ_3) Let $u(t,x)$ be scalar process represented by

$$du(t,x) = F(t,x,u,u_x,u_{xx})dt + C(t,x)f(t,x)dt + G(t,x,u)dw(t,x), \tag{2.14}$$

$$\text{I.C.} \quad u(0,x) = \phi(x), \quad x\varepsilon D$$

$$\text{B.C.} \quad u(t,x) = 0, \quad x\varepsilon\partial D,$$

where $f(t,x)$ is a control function to be specified, and for (2.14) the assumptions (C2.1)-(C2.5) in Sec.2.2 are made.

CHAPTER 3. LINEARIZATION METHODS

3.1. Introductory Remarks

In the theory of filtering and/or control of a class of nonlinear D.P.S., the approximation of the nonlinear function to some linear one will be expected to play a role as useful as in the lumped parameter systems. In this chapter, two possible methods of linearization based on the stochastic linearization and the Taylor series expansion are proposed. They are considered to be an extension of the idea of lumped parameter system to D.P.S.

3.2. Method by Taylor Series Expansion

Let us consider the system Σ_1 defined by Def.2.1, Sec.2.4, Chap.2. The nonlinear function $F(t,x,u,u_x,u_{xx})$ is expanded into a Taylor series around the $(\hat{u},\hat{u}_x,\hat{u}_{xx})$ as

$$(3.1) \qquad F(t,x,u,u_x,u_{xx}) = F(t,x,\hat{u},\hat{u}_x,\hat{u}_{xx}) + \left.\frac{\delta F}{\delta u}\right|_{\hat{u}}(u-\hat{u}) +$$

$$+ \frac{\delta F}{\delta u_x}\Big|_{\hat{u}_x}(u-\hat{u})_x + \frac{\delta F}{\delta u_{xx}}\Big|_{\hat{u}_{xx}}(u-\hat{u})_{xx} + \frac{1}{2}\frac{\delta^2 F}{\delta u^2}\Big|_{\hat{u}}(u-\hat{u})^2 + \cdots,$$

where $\hat{u}$, $\hat{u}_x$ and $\hat{u}_{xx}$ are the conditional expectations of u, u_x and u_{xx}, for more concretely these are defined in Chap.4, and $\delta F/\delta u$, $\delta F/\delta u_x, \cdots$ are the functional derivatives[152]. For simplicity, define a vector

$$v = [u_{xx}\ u_x\ u]', \tag{3.2}$$

and denote

$$F(t,x,u,u_x,u_{xx}) = F(t,x;v). \tag{3.3}$$

Then, for each $x \in D$, the Taylor series expansion (3.1) is represented as

$$F(t,x;v) = F(t,x;\hat{v}) + (v-\hat{v})'\frac{\delta F}{\delta v}\Big|_{\hat{v}} + \frac{1}{2}(v-\hat{v})'\frac{\delta^2 F}{\delta v^2}\Big|_{\hat{v}}(v-\hat{v}) + \cdots. \tag{3.4}$$

In (3.4), $\delta F/\delta v$ and $\delta^2 F/\delta v^2$ are a vector and a matrix with components $\{\delta F/\delta u,\ \delta F/\delta u_x,\ \delta F/\delta u_{xx}\}$ and $\{\delta^2 F/\delta u^2, \cdots\}$, and these will be given in Sec.4.4, Chap.4. This extension of Taylor series expansion will play a useful role in the nonlinear filtering theory in Chap.4.

3.3. Method by Stochastic Linearization

Consider the system Σ_3 defined by Def.2.3, Sec.2.4, Chap.2. By invoking the stochastic linearization technique reviewed in Chap.3, Part One, let us consider in this section an extension of the technique to the D.P.S.

Define a vector

$$v = [v_2'\ v_1'\ v_0]' \tag{3.5a}$$

with components

$$\begin{cases} v_0 = u \\ v_1 = [u_1\ u_2\ \cdots\ u_n]' \\ v_2 = [u_{11}\ u_{12}\ \cdots\ u_{1n}\ u_{22}\ \cdots\ u_{2n}\ \cdots\ u_{nn}]', \end{cases} \tag{3.5b}$$

where $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$ and n is the dimension of coordinate vector x. Then, for each $x \varepsilon D$, we expand the nonlinear function $\mathcal{F}(t,x;v)$ into

$$\mathcal{F}(t,x;v) = a(t,x) + B'(t,x)(v-\bar{v}) + e(t,x), \tag{3.6}$$

where $\bar{v}$ is a conditional expectation of v. The coefficients a and B are determined so as to minimize the conditional expectation of expansion error, i.e. $E\{|e(t,x)|^2|\phi(x)\}$. The procedure of the minimization is similar to that in Chap.3, Part One, and the results are

$$a(t,x) = E\{\mathcal{F}(t,x;v)|\phi(x)\} \triangleq \bar{\mathcal{F}}(t,x;v) \tag{3.7}$$

$$B(t,x) = S^{-1}(t,x)E\{(v-\bar{v})[\mathcal{F}(t,x;v)-\bar{\mathcal{F}}(t,x;v)]|\phi(x)\}, \tag{3.8}$$

where

$$S(t,x) = E\{(v-\bar{v})(v-\bar{v})'|\phi(x)\}. \tag{3.9}$$

The extension of stochastic linearization established here will be used in Chap.6 to obtain a feasible method of optimal control.

CHAPTER 4. STOCHASTIC ESTIMATION FOR NONLINEAR DISTRIBUTED PARAMETER SYSTEMS

4.1. Introductory Remarks

The estimation of states in noisy D.P.S. has important applications to identification, optimal and adaptive control as well as for systems described by ordinary differential equations. Many efforts have been done as previously surveyed in the subsection 1.1.A, Sec.1.1, Chap.1, for both linear and nonlinear D.P.S.

In this chapter, a general theory for filtering problems is developed for dynamical systems with the system noise of white Gaussian type and the boundary conditions and noisy observations which are made at the system output in the continuous time and spatial locations. Use is made of the theory of measure transformation established by Girsanov[45].

4.2. Preliminary Lemma

We consider, in this chapter, the mathematical models which is given by Σ_1 in Def.2.1, Sec.2.4, i.e.

$$\left.\begin{aligned} &(4.1) \qquad du(t,x) = F(t,x,u,u_x,u_{xx})dt + G(t,x,u)dw(t,x) \\ &(4.2) \qquad dy(t) = [\textstyle\int_D H(t,z,u)dz]dt + R(t)dv(t). \end{aligned}\right\}: \Sigma_1$$

The problem is to find the minimal variance estimate of the state $u(t,x)$ provided that the process $y(s)$ $(0\le s\le t)$ is obtained as the observed data.

In order to establish the filter dynamics via Radon-Nikodym derivative approach, a newly combined system is defined.

$$\left.\begin{aligned} &(4.3a) \qquad du(t,x) = F(t,x,u,u_x,u_{xx})dt + G(t,x,u)dw(t,x), \\ &(4.3b) \qquad u(0,x) = \phi(x), \\ &(4.4a) \qquad d\tilde{y}(t) = R(t)dv(t), \\ &(4.4b) \qquad \tilde{y}(0) = 0. \end{aligned}\right\}: \Sigma_0$$

Let μ_0 and μ_1 be the measures induced by the systems Σ_0 and Σ_1, respectively. The process $\{u(t,x), (t,x)\varepsilon R\}$ and the process $\{y(t), t\varepsilon[0,T]\}$ are mutually independent. Let $E_{(i)}\{\cdot|\mathcal{Y}_t\}$ denote the conditional expectation with respect to $\mu_i (i=0,1)$ conditioned by $\mathcal{Y}_t$, where the symbol $\mathcal{Y}_t$ denotes the minimal σ-algebra generated by $y(s)$ where $s\le t$. Let $C^{(1)}$ be the space of continuous functions on $[0,T]$ (for fixed $x\varepsilon D$). Let μ_u be the measure on the measurable space $(C^{(1+1)}, \mathcal{B}(u_t,y_0^t), \mu_1)$ for the system Σ_1, where the basic σ-algebra is the product σ-algebra $\mathcal{B}(u_t,y_0^t)=\mathcal{S}_t\times\mathcal{Y}_t$ and μ_1 is the product measure $\mu_1=\mu_u\times\mu_y$.

The systems Σ_1 and Σ_0 are respectively presented in a combined form:

$$(4.5) \qquad \Sigma_1: \quad d\begin{bmatrix} u(t,x) \\ y(t) \end{bmatrix} = \begin{bmatrix} F(t,x,u,u_x,u_{xx}) \\ \int_D H(t,z,u(t,z))dz \end{bmatrix}dt + \begin{bmatrix} G(t,x,u) & 0 \\ 0 & R(t) \end{bmatrix} d\begin{bmatrix} w(t,x) \\ v(t) \end{bmatrix}$$

$$(4.6) \qquad \Sigma_0: \quad d\begin{bmatrix} u(t,x) \\ \tilde{y}(t) \end{bmatrix} = \left\{\begin{bmatrix} F(t,x,u,u_x,u_{xx}) \\ \int_D H(t,z,u(t,z))dz \end{bmatrix} + \begin{bmatrix} G(t,x,u) & 0 \\ 0 & R(t) \end{bmatrix}\begin{bmatrix} 0 \\ -R^{-1}(t)\int_D H(t,z,u(t,z))dz \end{bmatrix}\right\}dt$$

$$+\begin{bmatrix} G(t,x,u) & 0 \\ 0 & R(t) \end{bmatrix} d\begin{bmatrix} w(t,x) \\ v(t) \end{bmatrix}.$$

For systems μ_0 and μ_1, we have the following lemma due to Girsanov.

Lemma 4.1. (Girsanov[45]) Let ζ_0^t be defined by

$$(4.7)\qquad \zeta_0^t = \int_0^t [0 \;\; -R^{-1}(s)\int_D H(s,z,u(s,z))dz]\, d\begin{bmatrix} w(s,x) \\ v(s) \end{bmatrix}$$

$$-\frac{1}{2}\int_0^t [0 \;\; -R^{-1}(s)\int_D H(s,z,u(s,z))dz]\begin{bmatrix} 0 \\ -R^{-1}(s)\int_D H(s,z,u(s,z))dz \end{bmatrix} ds.$$

Then, for the systems μ_0 and μ_1, the Radon-Nikodym derivative of μ_0 with respect to μ_1 is

$$(4.8)\qquad \frac{d\mu_0}{d\mu_1} = \exp\{\zeta_0^T\}.$$

4.3. Derivation of Filter Dynamics

In this section, we shall obtain a general version of the representation for the conditional expectation and prove that this version yields the optimal filtering process. Define

$$(4.9)\qquad \hat{u}(t,x) \triangleq E_{(1)}\{u(t,x)\,|\,\mathcal{Y}_t\},$$

and

$$(4.10)\qquad P(t,x,z) \triangleq E_{(1)}\{[u(t,x)-\hat{u}(t,x)][u(t,z)-\hat{u}(t,z)]\,|\,\mathcal{Y}_t\},$$

and also define the differential generator $\mathcal{G}$ of the diffusion process (4.1) as follows:

$$(4.11)\qquad \mathcal{G}V(t,u(t,x),u(t,z)) = \frac{\partial V(t,u(t,x),u(t,z))}{\partial t}$$

$$+ \frac{\delta V(t,u(t,x),u(t,z))}{\delta u(t,x)} \mathcal{F}(t,x,u(t,x),u_x,u_{xx})$$

$$+ \frac{\delta V(t,u(t,x),u(t,z))}{\delta u(t,z)} \mathcal{F}(t,z,u(t,z),u_z,u_{zz})$$

$$+ \frac{1}{2}\frac{\delta^2 V(t,u(t,x),u(t,z))}{\delta u^2(t,x)} G^2(t,x,u(t,x))Q(x,x) +$$

$$+ \frac{\delta^2 V(t,u(t,x),u(t,z))}{\delta u(t,x)\,\delta u(t,z)} G(t,x,u(t,x))Q(x,z)G(t,z,u(t,z))$$

$$+ \frac{1}{2} \frac{\delta^2 V(t,u(t,x),u(t,z))}{\delta u^2(t,z)} G^2(t,z,u(t,z))Q(z,z),$$

where V is a continuously twice differentiable function defined on the space $[0,T]\times R\times R$, and where $\delta V(t,u(t,x),u(t,z))/\delta u(t,x)$ denotes the partial derivative which is defined as the variation of V with respect to the function $u(t,x)$ at a point $x\varepsilon D$.

Theorem 4.1. Assume that the conditions (C2.1)-(C2.6) hold. Then there is a version of $E_{(1)}\{f(u(t,x),u(t,z))|Y_t\}$ which has the stochastic integro-differential,

$$(4.12) \qquad dE_{(1)}\{f(u(t,x),u(t,z))|Y_t\} = E_{(1)}\{Gf(u(t,x),u(t,z))|Y_t\}dt$$

$$+ [E_{(1)}\{f(u(t,x),u(t,z))[\textstyle\int_D H(t,\xi,u(t,\xi))d\xi]|Y_t\}$$

$$- E_{(1)}\{f(u(t,x),u(t,z))|Y_t\}E_{(1)}\{\textstyle\int_D H(t,\xi,u(t,\xi))d\xi|Y_t\}]$$

$$\times R^{-2}(t)[dy(t)-E_{(1)}\{\textstyle\int_D H(t,\xi,u(t,\xi))d\xi|Y_t\}dt],$$

for all $x,z\varepsilon D$, w.p.1, where $f(u(t,x),u(t,z))$ is a continuously twice differentiable function defined on the space $R\times R$.

Proof. Since μ_1 and μ_0 are equivalent, the derivative $d\mu_1/d\mu_0$ is obtained from (4.8) by using the relation,

$$(4.13) \qquad \Theta_0^T = \frac{d\mu_0}{d\mu_1} = \exp\{-\zeta_0^T\} \triangleq \exp\{\psi_0^T\},$$

where

$$(4.14) \qquad \psi_0^t = \textstyle\int_0^t [\int_D H(s,z,u(s,z))dz]R^{-2}(s)dy(s)$$

$$- \tfrac{1}{2}\textstyle\int_0^t [\int_D H(s,z,u(s,z))dz]^2 R^{-2}(s)ds.$$

Applying the Itô's formula to (4.13), it easily follows that [31,49]

$$(4.15) \qquad d\Theta_0^t = \Theta_0^t h_t R^{-2}(t)dy(t),$$

where the stochastic differential of (2.5) has been used.

Let $f(\cdot,\cdot)$ be any scalar Baire function such that $E_{(1)}\{|f(u(t,x), u(t,z))|\}<\infty$ for all $x,z\in D$. Then it follows that (see Loève[90] and Zakai [165])

$$(4.16) \qquad E_{(1)}\{f(u(t,x),u(t,z))|\mathcal{Y}_t\} = \frac{E_{(0)}\{f(u(t,x),u(t,z))\Theta_0^t|\mathcal{Y}_t\}}{E_{(0)}\{\Theta_0^t|\mathcal{Y}_t\}}.$$

Express the right-hand side of (4.16) by

$$(4.17) \qquad V(F_t,B_t) \triangleq \frac{F_t}{B_t},$$

where

$$(4.18) \qquad F_t \triangleq E_{(0)}\{f(u(t,x),u(t,z))\exp\{\psi_0^t\}|\mathcal{Y}_t\},$$

$$(4.19) \qquad B_t \triangleq E_{(0)}\{\exp\{\psi_0^t\}|\mathcal{Y}_t\}.$$

Then, it follows that

$$(4.20) \qquad dV(F_t,B_t) \cong \frac{dF_t}{B_t} - \frac{F_t}{B_t^{\,2}}(dB_t) - \frac{(dF_t)(dB_t)}{B_t^{\,2}} + \frac{F_t}{B_t^{\,3}}(dB_t)^2.$$

Using the relations (4.15), (4.18), (4.19) and the differential generator defined by (4.11), we have

$$(4.21) \qquad dF_t = E_{(0)}\{\mathcal{G}f(u(t,x),u(t,z))\exp\{\psi_0^t\}|\mathcal{Y}_t\}dt$$
$$+ E_{(0)}\{f(u(t,x),u(t,z))\exp\{\psi_0^t\}h_tR^{-2}(t)|\mathcal{Y}_t\}dy(t),$$

$$(4.22) \qquad dB_t = E_{(0)}\{\exp\{\psi_0^t\}h_tR^{-2}(t)|\mathcal{Y}_t\}dy(t),$$

$$(4.23) \qquad (dF_t)(dB_t) = E_{(0)}\{f(u(t,x),u(t,z))\exp\{\psi_0^t\}h_t|\mathcal{Y}_t\}R^{-2}(t)$$
$$\times E_{(0)}\{\exp\{\psi_0^t\}h_t|\mathcal{Y}_t\}dt,$$

$$(4.24) \qquad (dB_t)^2 = E_{(0)}\{\exp\{\psi_0^t\}h_t|\mathcal{Y}_t\}R^{-2}(t)E_{(0)}\{\exp\{\psi_0^t\}h_t|\mathcal{Y}_t\}dt.$$

Substitution of (4.21) to (4.24) into (4.20) completes the proof.

Theorem 4.2. Assume that the same conditions as in Theorem 4.1 hold. Then the optimal estimate of $u(t,x)$ is determined by the following stochastic integro-differential equation,

$$(4.25)\quad \begin{aligned} d\hat{u}(t,x) &= E_{(1)}\{F(t,x,u,u_x,u_{xx})|Y_t\}dt \\ &+ [E_{(1)}\{u(t,x)[\int_D H(t,z,u(t,z))dz]|Y_t\} \\ &- E_{(1)}\{u(t,x)|Y_t\}E_{(1)}\{\int_D H(t,z,u(t,z))dz|Y_t\}]R^{-2}(t) \\ &\times[dy(t)-E_{(1)}\{\int_D H(t,z,u(t,z))dz|Y_t\}dt], \quad \text{w.p.1.} \end{aligned}$$

Proof. In Theorem 4.1, set as $f(u(t,x),u(t,z))\equiv u(t,x)$. Then (4.25) is obtained, because

$$(4.26)\quad Gu(t,x) = F(t,x,u,u_x,u_{xx}).$$

Corollary 4.1. Suppose that the mathematical models of both the system and the observation mechanism are respectively described by the linear stochastic differential equation and the linear integro-differential equation, i.e.

$$(4.27)\quad F(t,x,u(t,x),u_x,u_{xx}) \equiv L_x u(t,x),$$

$$(4.28)\quad \int_D H(t,z,u(t,z))dz \equiv \int_D H(t,z)u(t,z)dz,$$

$$(4.29)\quad G(t,x,u(t,x)) \equiv G(t,x),$$

where L_x is an elliptic operator.

The optimal filter dynamics and the associated error covariance equation are respectively given as follows:

$$(4.30)\quad \begin{aligned} d\hat{u}(t,x) &= L_x\hat{u}(t,x)dt \\ &+ [\int_D H(t,\xi)P(t,x,\xi)d\xi]R^{-2}(t)\{dy(t)-[\int_D H(t,\xi)\hat{u}(t,\xi)d\xi]dt\} \end{aligned}$$

and

$$(4.31)\quad \frac{\partial P(t,x,z)}{\partial t} = (L_x+L_z)P(t,x,z) + G(t,x)Q(x,z)G(t,z) -$$

$$- [\int_D H(t,\xi)P(t,x,\xi)d\xi]R^{-2}(t)[\int_D H(t,\xi)P(t,\xi,z)d\xi]$$

and, $u(t,x)$, $L_x u(t,x)$, $P(t,x,z)$, $L_x P(t,x,z)$ and $L_z P(t,x,z)$ tend to 0 as $x \to \partial D$.

Proof. The proof is straightforward as shown in Appendix A. Equations (4.30) and (4.31) coincide with the results of [82].

4.4. Approximate Filter Dynamics

The filter dynamics derived in the previous section reveals that an exact realization of optimal nonlinear filters requires infinite dimensional stochastic moments, which are practically impossible.

In this section, the author presents a possible method of approximation to a realizable filter by means of the local expansion of nonlinear functions.

Define a new vector which was introduced in Sec.3.2,

$$v = [u_{xx} \quad u_x \quad u]', \tag{4.32}$$

and denote

$$F(t,x,u(t,x),u_x,u_{xx}) \triangleq F(t,x;v) \tag{4.33}$$

$$\hat{v} = [\hat{u}_{xx} \quad \hat{u}_x \quad \hat{u}]', \tag{4.34}$$

where the symbol "$\hat{\cdot}$" denotes $E_{(1)}\{\cdot|Y_t\}$.

Expanding the nonlinear function $F(t,x,u,u_x,u_{xx})$ into a Taylor series, we have

$$F(t,x;v) \cong F(t,x;\hat{v}) + (v-\hat{v})'\frac{\delta F}{\delta v}\Big|_{\hat{v}} + \frac{1}{2}(v-\hat{v})'\frac{\delta^2 F}{\delta v^2}\Big|_{\hat{v}}(v-\hat{v}), \tag{4.35}$$

where

$$\frac{\delta F}{\delta v} = [\frac{\delta F}{\delta u_{xx}} \quad \frac{\delta F}{\delta u_x} \quad \frac{\delta F}{\delta u}]', \tag{4.36}$$

and where $\delta F/\delta v$ denotes the vector partial derivative, and

$$(4.37)\qquad \frac{\delta^2 F}{\delta v^2} = \begin{bmatrix} \frac{\delta^2 F}{\delta u_{xx}^2} & \frac{\delta^2 F}{\delta u_{xx}\delta u_x} & \frac{\delta^2 F}{\delta u_{xx}\delta u} \\ \frac{\delta^2 F}{\delta u_x \delta u_{xx}} & \frac{\delta^2 F}{\delta u_x^2} & \frac{\delta^2 F}{\delta u_x \delta u} \\ \frac{\delta^2 F}{\delta u \delta u_{xx}} & \frac{\delta^2 F}{\delta u \delta u_x} & \frac{\delta^2 F}{\delta u^2} \end{bmatrix}.$$

Then it follows that

$$(4.38)\qquad \hat{F}(t,x;v) = F(t,x;\hat{v}) + \frac{1}{2}\,\mathrm{tr.}\{S(t,x)\frac{\delta^2 F}{\delta v^2}|_{\hat{v}}\},$$

where

$$(4.39)\qquad S(t,x) \triangleq E_{(1)}\{(v-\hat{v})(v-\hat{v})'|Y_t\}.$$

Similarly, since the functions $H(t,x,u)$ and $G(t,x,u)$ are respectively approximated by

$$(4.40)\qquad H(t,x,u) = H(t,x,\hat{u}) + (u-\hat{u})\frac{\delta H}{\delta u}|_{\hat{u}} + \frac{1}{2}(u-\hat{u})^2\frac{\delta^2 H}{\delta u^2}|_{\hat{u}}$$

and

$$(4.41)\qquad G(t,x,u) = G(t,x,\hat{u}) + (u-\hat{u})\frac{\delta G}{\delta u}|_{\hat{u}} + \frac{1}{2}(u-\hat{u})^2\frac{\delta^2 G}{\delta u^2}|_{\hat{u}},$$

then we have

$$(4.42)\qquad \hat{H}(t,x,u) = H(t,x,\hat{u}) + \frac{1}{2}\frac{\delta^2 H}{\delta u^2}|_{\hat{u}}P(t,x,x),$$

$$(4.43)\qquad \hat{G}(t,x,u) = G(t,x,\hat{u}) + \frac{1}{2}\frac{\delta^2 G}{\delta u^2}|_{\hat{u}}P(t,x,x).$$

Substituting (4.38) and (4.40) into (4.25) and using (4.42), we have

$$(4.44)\qquad d\hat{u}(t,x) = [F(t,x;\hat{v}) + \frac{1}{2}\,\mathrm{tr.}\{S(t,x)\frac{\delta^2 F}{\delta v^2}|_{\hat{v}}\}]dt$$

$$+ [\int_D \frac{\delta H}{\delta u}|_{\hat{u}z}\, P(t,x,z)\,dz]R^{-2}(t)\{dy(t)-[\int_D (H(t,x,\hat{u})+$$

$$+ \frac{1}{2} \frac{\delta^2 H}{\delta u^2}\Big|_{\hat{u}^z} P(t,z,z))dz]dt\},$$

where superscripts denote the spatial point. The covariance equation is obtained through a simple calculation by substituting (4.35), (4.38), (4.40)-(4.43) into (A.6) (see Appendix A). The result is*

$$(4.45) \qquad dP(t,x,z) = [\frac{\delta F^x}{\delta u_{xx}}|_{\hat{u}^x} \frac{\partial^2}{\partial x^2}P(t,x,z) + \frac{\delta F^x}{\delta u_x}|_{\hat{u}^x} \frac{\partial}{\partial x}P(t,x,z)$$

$$+ \frac{\delta F^x}{\delta u}|_{\hat{u}^x} P(t,x,z) + \frac{\delta F^z}{\delta u_{zz}}|_{\hat{u}^z} \frac{\partial^2}{\partial z^2}P(t,x,z) + \frac{\delta F^z}{\delta u_z}|_{\hat{u}^z} \frac{\partial}{\partial z}P(t,x,z)$$

$$+ \frac{\delta F^z}{\delta u}|_{\hat{u}^z} P(t,x,z)]dt + [G(t,x,u)G(t,z,u)$$

$$+ \frac{\delta G^x}{\delta u}|_{\hat{u}^x} \frac{\delta G^z}{\delta u}|_{\hat{u}^z} P(t,x,z) + \frac{1}{2}G(t,z,u)\frac{\delta^2 G^x}{\delta u^2}|_{\hat{u}^x} P(t,x,x)$$

$$+ \frac{1}{2}G(t,x,u)\frac{\delta^2 G^z}{\delta u^2}|_{\hat{u}^z} P(t,z,z)]Q(x,z)dt$$

$$- \frac{1}{2}P(t,x,z)[\int_D \frac{\delta^2 H^\xi}{\delta u^2}|_{\hat{u}^\xi} P(t,\xi,\xi)d\xi]R^{-2}(t)[dy(t)$$

$$-\{\int_D(H(t,\xi,\hat{u}) + \frac{1}{2} \frac{\delta^2 H^\xi}{\delta u^2}|_{\hat{u}^\xi} P(t,\xi,\xi))d\xi\}dt]$$

$$- [\int_D \frac{\delta H^\xi}{\delta u}|_{\hat{u}^\xi} P(t,x,\xi)d\xi]R^{-2}(t)[\int_D \frac{\delta H^\xi}{\delta u}|_{\hat{u}^\xi} P(t,\xi,z)d\xi]dt,$$

where we assumed that [111]

$$(4.46) \qquad E_{(1)}\{(u^x-\hat{u}^x)(u^z-\hat{u}^z)(h_t-\hat{h}_t)|Y_t\} =$$

---

* In (4.45), the first and the second terms in the right-hand side should be interpreted respectively by the more precise expressions as

$$\sum_{i,j=1}^{n} \frac{\delta F^x}{\delta u_{x_i x_j}}\Big|_{\hat{u}^x} \frac{\partial^2}{\partial x_i \partial x_j}P(t,x,z) \text{ and } \sum_{j=1}^{n} \frac{\delta F^x}{\delta u_{x_j}}\Big|_{\hat{u}^x} \frac{\partial}{\partial x_j}P(t,x,z).$$

$$= -\frac{1}{2} P(t,x,z)[\int_D \frac{\delta^2 H}{\delta u^2}\Big|_{\hat{u}\xi} P(t,\xi,\xi)d\xi].$$

Equations (4.44) and (4.45) describe the approximated dynamic structure for generating the current estimates $\hat{u}(t,x)$ with the given initial conditions, $\hat{u}(0,x)=E_{(1)}\{u(0,x)\}$, $P(0,x,z)=E_{(1)}\{[u(0,x)-u(0,x)]\cdot[u(0,z)-u(0,z)]\}$, and the given boundary conditions.

4.5. An Illustrative Example and Digital Simulations

For the purpose of exploring the quantitative aspects, we shall consider the following scalar nonlinear stochastic diffusion systems:

$$(4.47)\qquad \begin{cases} du(t,x) = [\dfrac{\partial^2 u(t,x)}{\partial x^2} + \beta u^2(t,x)]dt + Gdw(t,x) \\ u(0,x) = A\sin^2 \pi x, \quad 0\leq x\leq 1 \\ u(t,x) = 0 \quad \text{on } x=0,\ 1 \end{cases}$$

$$(4.48)\qquad \begin{cases} dy(t) = [\int_0^1 Hu(t,z)dz]dt + Rdv(t) \\ y(0) = 0, \end{cases}$$

where β, G, A, H and R are constants, and the variance of the Brownian motion process $w(t,x)$ is assumed to be

$$(4.49)\qquad Q(x,z) = \delta(x-z), \quad 0<x,z<1.$$

From (4.44) and (4.45), the approximate filter dynamics is determined by

$$(4.50)\qquad d\hat{u}(t,x) = [\frac{\partial^2 \hat{u}(t,x)}{\partial x^2} + \beta\hat{u}^2(t,x) + \beta P(t,x,x)]dt$$
$$+ [\int_0^1 HP(t,x,\xi)d\xi]R^{-2}\{dy(t)-[\int_0^1 Hu(t,\xi)d\xi]dt\},$$

with the covariance equation,

$$(4.51)\qquad \frac{\partial P(t,x,z)}{\partial t} = \frac{\partial^2}{\partial x^2}P(t,x,z) + 2\beta u(t,x)P(t,x,z)$$
$$+ \frac{\partial^2}{\partial z^2}P(t,x,z) + 2\beta u(t,z)P(t,x,z) + G^2 Q(x,z) -$$

$$- [\int_0^1 HP(t,x,\xi)d\xi]R^{-2}[\int_0^1 HP(t,\xi,z)d\xi].$$

Equations (4.47) to (4.51) are simulated on a digital computer with a subroutine for the generation of random disturbances, $w(t,x)$ and $v(t)$.

Suppose that observations are taken at discrete time t_j, and that $\delta t_j = t_{j+1} - t_j$ $(j=0,1,2,\cdots)$ where δt_j is sufficiently short. The observation, δy_j, can be taken to be

$$(4.52) \qquad \delta y_j = y(t_{j+1}) - y(t_j)$$

$$\cong [\sum_{i=0}^{N-1} Hu(t_j,x_i)\delta x_i]\delta t_j + R\delta v_j,$$

where the spatial interval [0,1] is divided into N partitions such that $\delta x_i = x_{i+1} - x_i$ $(i=0,1,2,\cdots,N-1)$, and

$$(4.53) \qquad \delta v_j \cong v(t_{j+1}) - v(t_j).$$

Define the standard difference operators D_+, D_- and D_0 in the usual way, i.e. (e.g. see [41])

$$D_+u(t,x_i) = \frac{u(t,x_{i+1}) - u(t,x_i)}{\delta x_i},$$

$$D_-u(t,x_i) = \frac{u(t,x_i) - u(t,x_{i-1})}{\delta x_{i-1}},$$

$$D_0u(t,x_i) = \frac{u(t,x_{i+1}) - u(t,x_{i-1})}{\delta x_i + \delta x_{i-1}}.$$

The increment of the state u at the point x_i is

$$(4.54) \qquad \delta u_j(x_i) = u(t_{j+1},x_i) - u(t_j,x_i)$$

$$\cong [D_+D_-u(t_j,x_i)+\beta u^2(t_j,x_i)]\delta t_j + G\delta w_j(x_i),$$

where

$$(4.55) \qquad \delta w_j(x_i) \cong w(t_{j+1},x_i) - w(t_j,x_i).$$

Recall that increments of the Brownian motion processes, $\delta w_j(x_i)$ and δv_j, are respectively approximated by $\delta w_j(x_i) \cong n_i(t_j)\sqrt{\delta t_j}$ and $\delta v_j \cong n_v(t_j)\sqrt{\delta t_j}$, where $n_i(t_j)$ and $n_v(t_j)$ are mutually independent Gaussian random numbers with N[0,1] (see Sec.6.7, Chap.6 in Part One). Then (4.52) and (4.54) may respectively be computed by

$$\delta y_j \cong [\sum_{k=0}^{N-1} Hu(t_j,x_k)\delta x_k]\delta t_j + Rn_v(t_j)\sqrt{\delta t_j} \tag{4.56}$$

$$u_j(x_i) \cong [D_+D_-u(t_j,x_i)+\beta u^2(t_j,x_i)]\delta t_j + Gn_i(t_j)\sqrt{\delta t_j}. \tag{4.57}$$

Simple calculations show that (4.50) and (4.51) are also approximated by

$$\begin{aligned}\hat{u}(t_{j+1},x_i) \cong \hat{u}(t_j,x_i) &+ [D_+D_-\hat{u}(t_j,x_i)+\beta\hat{u}^2(t_j,x_i) \\ &\qquad +\beta P(t_j,x_i,x_i)]\delta t_j \\ &+ [\sum_{k=0}^{N-1} HP(t_j,x_i,x_k)\delta x_k]R^{-2}\{\delta y_j-[\sum_{k=0}^{N-1} H\hat{u}(t_j,x_k)\delta x_k]\delta t_j\}\end{aligned} \tag{4.58}$$

$$\begin{aligned}P(t_{j+1},x_i,x_\nu) &\cong P(t_j,x_i,x_\nu) \\ &+ [(D_+D_-)_{x_i}P(t_j,x_i,x_\nu)+2\beta\hat{u}(t_j,x_i)P(t_j,x_i,x_\nu)]\delta t_j \\ &+ [(D_+D_-)_{x_\nu}P(t_j,x_i,x_\nu)+2\beta\hat{u}(t_j,x_\nu)P(t_j,x_i,x_\nu)]\delta t_j \\ &+ G^2Q(x_i,x_\nu)\delta t_j \\ &- [\sum_{k=0}^{N-1} HP(t_j,x_i,x_k)\delta x_k]R^{-2}[\sum_{k=0}^{N-1} HP(t_j,x_k,x_\nu)\delta x_k]\delta t_j, \\ &\qquad (\nu=0,1,2,\cdots,N).\end{aligned} \tag{4.59}$$

where the operator $(D_+D_-)_{x_i}$ denotes the operation at the spatial point x_i. Letting $j=0,1,2,\cdots$, equations (4.52) to (4.59) are simulated on a digital computer to obtain the running values of $\hat{u}(t_j,x_i)$ and $P(t_j,x_i,x_\nu)$ with a set of preassigned initial data.

Figures 4.1(a) and 4.1(b) show the bird's-eye views of the states of the true system $u(t,x)$ and the estimation $\hat{u}(t,x)$. Naturally, although the true solution process can not be observed in practice, this is shown only for convenience of discussions. In the digital simulations, δx_i and δt_j

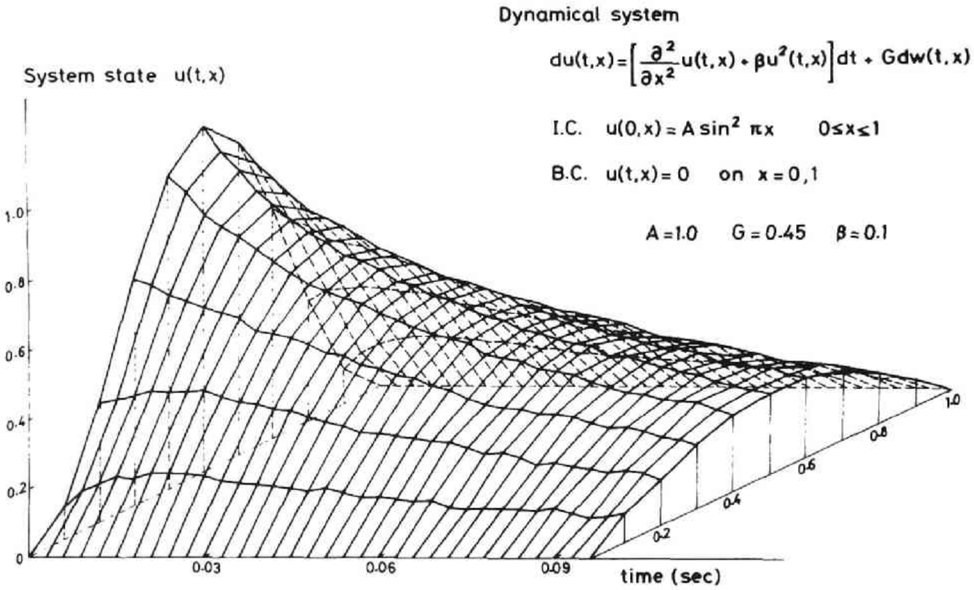


Fig.4.1(a). True system states u(t,x).

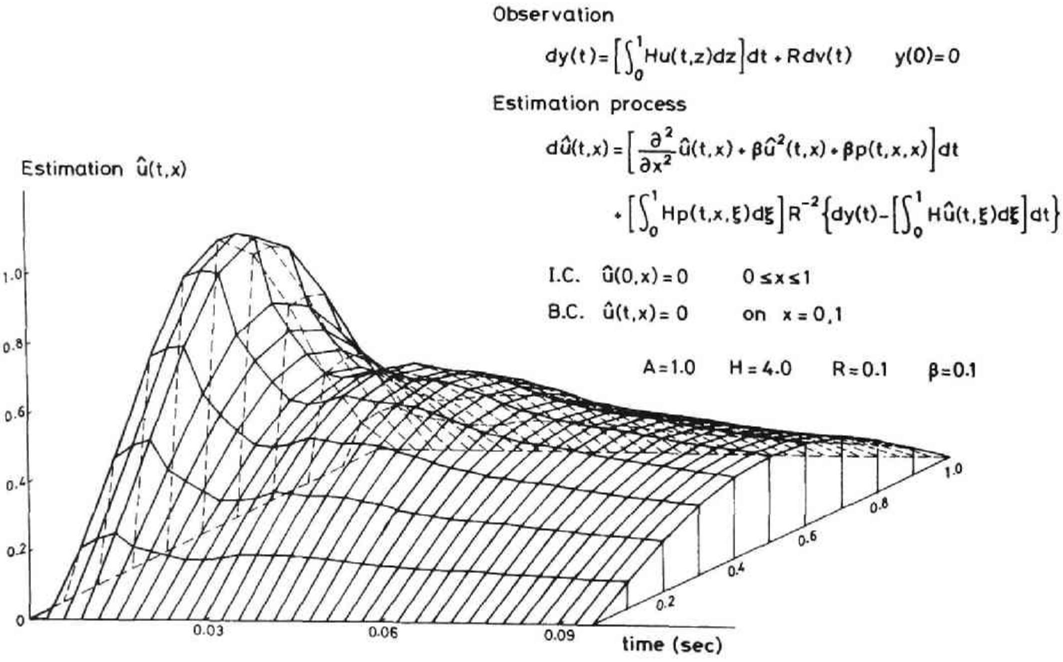


Fig.4.1(b). Filtered estimate $\hat{u}(t,x)$.

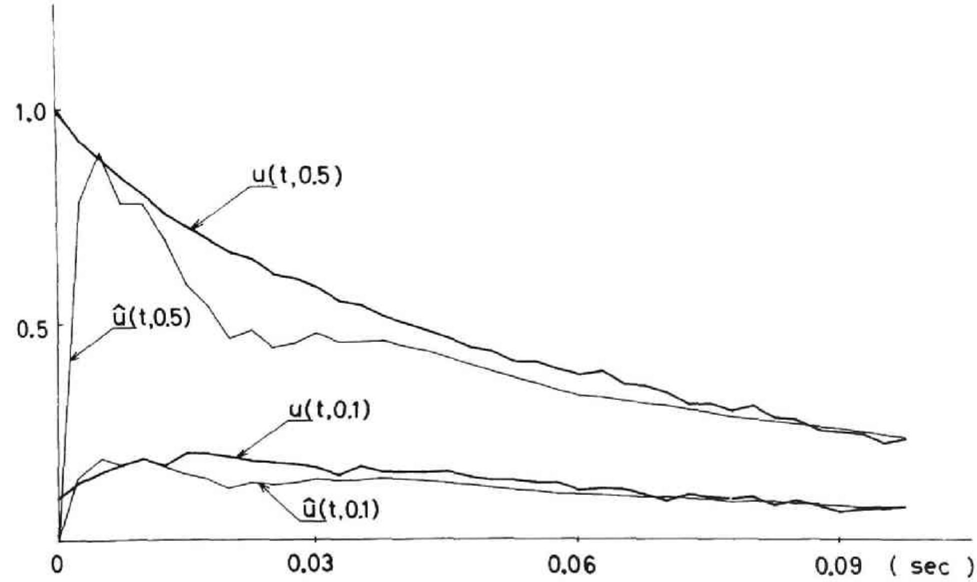


Fig.4.2. Comparison of true and filtering processes at x=0.1 and x=0.5.

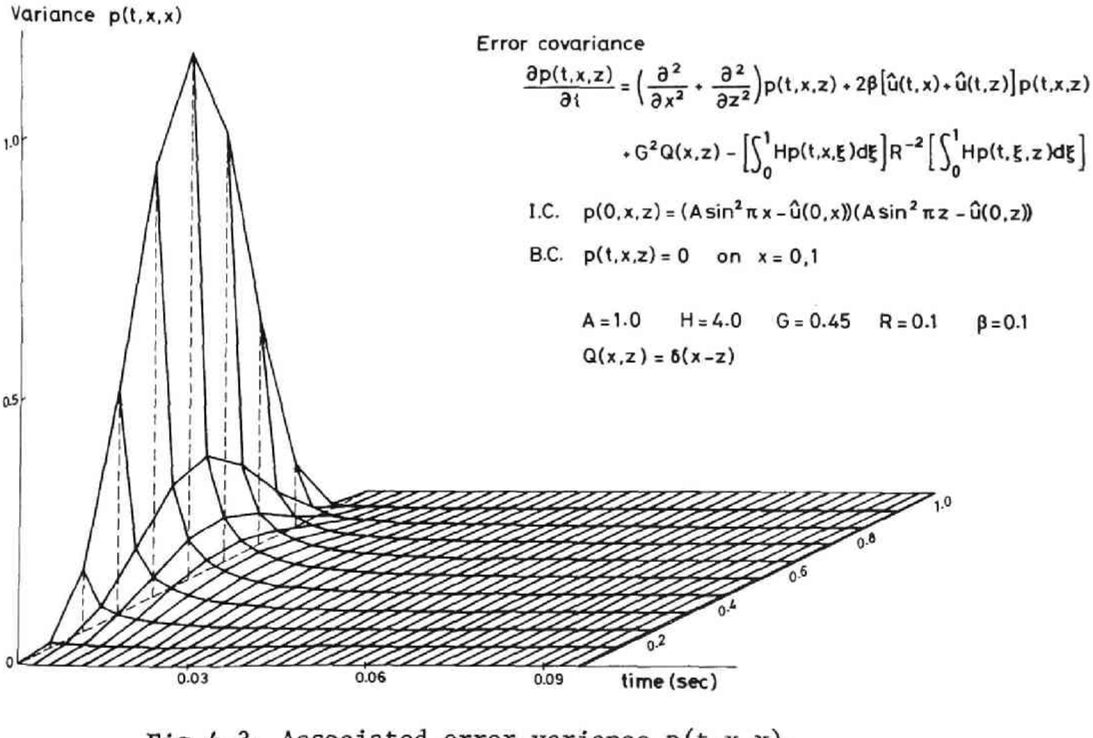


Fig.4.3. Associated error variance p(t,x,x).

are both equi-partitions, 0.02 and 0.0001 (sec) respectively. The sample values are depicted in 0.1 and 0.0015 (sec) intervals with respect to x and t. Values of each of the parameters and the initial conditions are also shown in the figures. Figure 4.2 shows the convergence of the approximate filter at spatial locations x=0.1 and x=0.5. From the figure it can be seen that the rate of convergence is rapid at respective locations. Figure 4.3 shows a numerical aspect of associated error variance, $P(t,x,x)$.

Although it is extremely difficult to justify analytically the accuracy of the proposed technique, experimental results obtained reveal that the approximate filter based on the second-order expansion shows good performance and will play a useful role to the realization of a broad class of stochastic nonlinear D.P.S.

4.6. Discussions and Summary

In this chapter, the estimation problem has been solved for a general class of nonlinear D.P.S. In particular, the Radon-Nikodym derivative approach has been employed to derive the version of representations for conditional expectation. The result reveals that the optimal estimation is generated by the solution of a stochastic integro-differential equation. If both the system and the observation mechanism are linear, the estimation equation coincides with that obtained in [82].

CHAPTER 5. PARAMETER IDENTIFICATION FOR LINEAR DISTRIBUTED PARAMETER SYSTEMS

5.1. Introductory Remarks

Physical processes which may be modeled by a class of linear or non-linear partial differential equations involve such real physical systems as heat exchangers, chemical reactors, nuclear reactors and environmental systems. It is a usual way that a given physical process can be specified by the basic conservation principles via constitutive relations. We know that many serious problems in real physical systems were solved formerly without a complete understanding of relevant physical and/or biological factors. However, pragmatic approaches to the solution of problems can be adopted only when the cause-effect relations are readily apparent.

In the most cases, unknown parameters appear in the models and these must be identified by comparing experimental measurements of the process and the solutions to the equations describing the process. The unknown parameters are seems to be surely constant or can be assumed constant over an interesting range.

Several trials have recently been made on the parameter identification for D.P.S. as surveyed in the subsection 1.1.B, Sec.1.1, Chap.1. In this chapter, a new method of parameter identification is presented by invoking the Bayesian theoretic approach.

Let the mathematical models of the physical system and the observation mechanism be given by Σ_2 in Def.2.2, Sec.2.4, i.e.

$$\left.\begin{aligned} &(5.1) \qquad du(t,x) = L_x u(t,x)dt + g(t,x,\theta)dt + G(t,x)dw(t,x) \\ &(5.2) \qquad dy(t) = [\int_D H(t,z)u(t,z)dz]dt + R(t)dv(t). \end{aligned}\right\} : \Sigma_2$$

The work presented in this section was motivated by actual air pollution problems in environmental systems. The mathematical model given by (5.1) is a somewhat simplified diffusion model of air pollution. The state of the problems is characterized by the scalar state $u(t,x)$ which is considered, for example, as the field of temperature or density of the air pollutant. The first term on the right-hand side is due to diffusion; the second term is the representation of the air pollution source term; and the final term represents the additive system noise caused by the environmental noise. From the viewpoint of air pollution prevention, the unknown parameter θ expresses the pollution source to be identified. The objectives are twofold: (i) to solve some specific identification problems, and (ii) to derive systematic method for establishing the parameter identification and state estimation algorithm under noisy measurements.

In a practical problem, we also have an additional problem of determining the forcing term g and the coefficient of the model. An example of identifying the function g is

$$(5.3) \qquad g(t,x,\theta) = \sum_{i=1}^{N} C_i(t)\delta(x-\theta^{(i)}),$$

where $C_i(t)$ is a known function expressing the intensity of the i-th pollution source and δ is a Dirac delta function. Naturally, if we adopt the model given by (5.3), then there exists a violation of the mathematical conditions for the existence of the solution to (5.1). However, since the mathematical aspect will be discussed elsewhere, we will not go into any details on the existing problem of solutions in this chapter.* In this situation, it is of great interest to derive the identification methods for the unknown parameters $\theta^{(i)}$. To fix the ideas, we shall first consider the case where N=1, because the extension to N forcing terms is straightforward with few changes. Thus with (5.3), equation (5.1) is written as

$$du(t,x) = L_x u(t,x)dt + C(t)\delta(x-\theta)dt + G(t,x)dw(t,x), \quad \text{for } t\varepsilon[0,T],\ x\varepsilon D, \tag{5.4}$$

with associated initial and boundary conditions,

$$u(0,x) = \phi(x), \quad x\varepsilon D, \tag{5.5a}$$

$$u(t,x) = 0, \quad t\varepsilon[0,T],\ x\varepsilon\partial D, \tag{5.5b}$$

where $\phi(x)$ is the known initial condition on u.

Although in most practical cases, changes of admissible values of θ are continuous with the *a priori* probability $P(\theta)$, as might be expected, the computational requirements are in general excessive. Consequently, the *a priori* probability $P(\theta)$ is assumed to be

$$P(\theta) = \sum_{i=1}^{K} P(\theta_i)\delta(\theta-\theta_i), \tag{5.6}$$

that is, the parameter θ changes over the finite set of points θ_1, $\theta_2, \cdots$, θ_K.

The choice of the mathematical model (5.2) implies the situation in which observations are continuously made on the system state with respect to time and spatial points. This is only for mathematical convenience to develop the theoretical aspect in the continuous parameter process. A more practical model will be taken into account later.

Let Y_t be the observation data up to the present time t. The problem is to find the best estimate of the unknown parameter θ and the system state u(t,x) based on the observed data sequence $\{y(s),\ 0\leq s\leq t\}$.

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* The rigorous proof of existence and uniqueness of the solution to (5.1) requires the knowledges of generalized random field and distribution theory [43,110].

5.2. Preliminary Lemmas

Let $A^{(i)}$ be the event such that

$$(5.7)\qquad A^{(i)} = \{\omega:\ \theta(\omega)=\theta_i\},\quad (i=1,2,\cdots,K),$$

where ω is the generic point of the probability space Ω.

With the event $A^{(i)}$, equation (5.2) can be expressed by

$$(5.8)\qquad A^{(i)}:\ dy(t) = [\int_D H(t,z)u_i(t,z)dz]dt + R(t)dv(t),$$

where $u_i(t,x)$ is the solution of

$$(5.9)\qquad du_i(t,x) = L_x u_i(t,x)dt + C(t)\delta(x-\theta_i)dt + G(t,x)dw(t,x),$$

with the associated initial and boundary conditions

$$(5.10a)\qquad u_i(0,x) = \phi(x),\quad x\varepsilon D,$$

$$(5.10b)\qquad u_i(t,x) = 0,\quad x\varepsilon\partial D,\quad t\varepsilon[0,T].$$

Let P_i $(i=1,2,\cdots,K)$ and P_0 denote respectively the measures induced in the space of continuous functions by the observation $\{y(s),\ 0\leq s\leq t\}$ under $A^{(i)}$ and by the observation

$$(5.11a)\qquad A_0:\ d\tilde{y}(t) = R(t)dv(t),$$

$$(5.11b)\qquad \tilde{y}(0) = 0.$$

Then, we have the following lemma.

<u>Lemma 5.1.</u> [58,128,138] Let P_i and P_0 be the two measures induced by (5.8) and (5.11) respectively. Then, it follows that

(1) $P_i \ll P_0$, that is, P_i is absolutely continuous with respect to P_0; [118]

(2) the Radon-Nikodym derivative of P_i with respect to P_0 is given by

$$(5.12)\qquad \frac{dP_i}{dP_0} = \exp\{\int_0^T \hat{h}_i(t,u_t)R^{-2}(t)dy(t) - \frac{1}{2}\int_0^T \hat{h}_i^{\,2}(t,u_t)R^{-2}(t)dt\},$$

where u_t is a sample process at a fixed x and

$$(5.13)\qquad \hat{h}_i(t,u_t) \triangleq E\{\int_D H(t,x)u(t,x)dx|\mathcal{Y}_t,\ A^{(i)}\}$$

$$= \int_D H(t,x)\hat{u}_i(t,x)dx$$

and $\hat{u}_i(t,x)$ are defined by

$$(5.14)\qquad \hat{u}_i(t,x) = E\{u(t,x)|\mathcal{Y}_t,\ A^{(i)}\},\quad (i=1,2,\cdots,K)$$

and these are determined by the solution processes of the filtering equations given by the following lemma.

Lemma 5.2. Assume that the conditions (C2.4) and (C2.7)-(C2.10) hold. Then the optimal estimates $\hat{u}_i(t,x)$ of the system state $u(t,x)$ under $A^{(i)}$ are determined by

$$(5.15)\qquad d\hat{u}_i(t,x) = L_x\hat{u}_i(t,x)dt + C(t)\delta(x-\theta_i)dt$$

$$+ [\int_D H(t,\xi)S_i(t,\xi,x)d\xi]R^{-2}(t)\{dy(t)-[\int_D H(t,\xi)\hat{u}_i(t,\xi)d\xi]dt\ ,$$

$$(i=1,2,\cdots,K),$$

where $S_i(t,\xi,x)$ is the associated covariance defined by

$$(5.16)\qquad S_i(t,\xi,x) = E\{[u(t,\xi)-\hat{u}_i(t,\xi)][u(t,x)-\hat{u}_i(t,x)]|\mathcal{Y}_t,\ A^{(i)}\}$$

and this is determined by

$$(5.17)\qquad \frac{\partial}{\partial t} S_i(t,\xi,x) = (L_\xi+L_x)S_i(t,\xi,x) + G(t,\xi)Q(\xi,x)G(t,x)$$

$$- [\int_D H(t,z)S_i(t,z,\xi)dz]R^{-2}(t)[\int_D H(t,z)S_i(t,z,x)dz].$$

The proof may easily be completed as a direct consequence of [138] or [82].

Remark 5.1: If, for the preassigned initial values, the relations $\hat{u}_i(0,x)=\hat{u}_0(x)$, $S_i(0,\xi,x)=S_0(\xi,x)$ hold for all i, then it follows from (5.17) that $S_i(t,\xi,x)=S_j(t,\xi,x)$ for all i and j.

Remark 2.2: Version of the likelihood-ratio: It is readily understood that the Radon-Nikodym derivative, (5.12), in Lemma 5.1 is rewritten by

$$(5.18)\qquad \frac{dP_i}{dP_0} = \frac{p\{\mathcal{Y}_T|A^{(i)}\}}{p\{\mathcal{Y}_T|A_0\}} \triangleq \Lambda_i(T),$$

where $\Lambda_i(T)$ is the likelihood-ratio.

By applying Itô stochastic calculus to (5.12) and (5.18), we have the following lemma.

Lemma 5.3. A sample process of the likelihood-ratio function $\Lambda_i(t)$ $(i=1,2,\cdots,K)$ is determined by the following stochastic differential equation,

$$d\Lambda_i(t) = \Lambda_i(t)\hat{h}_i(t,u_t)R^{-2}(t)dy(t), \tag{5.19a}$$

$$\Lambda_i(0) = 1. \tag{5.19b}$$

The proof is shown in Appendix B.

Lemma 5.4. Let β be a vector defined by $\beta=[u\ \theta']'$, where " ' " denotes its transpose. The minimal conditional mean square performance criterion,

$$J(\hat{\beta}) \triangleq E\{[\beta(t,x)-\hat{\beta}(t,x)]'[\beta(t,x)-\hat{\beta}(t,x)]|\mathcal{Y}_t\}, \tag{5.20}$$

is reduced to

$$J(\hat{\theta},\hat{u}) = E\{\|\theta-\hat{\theta}(t)\|^2|\mathcal{Y}_t\} + E\{[u(t,x)-\hat{u}(t,x)]^2|\mathcal{Y}_t\}, \tag{5.21}$$

where the symbol "$\|\cdot\|$" expresses the Euclidean norm.

5.3. Parameter Identification

According to Lemma 5.4, the minimal conditional mean square performance criterion (5.21) is used here, for which the conditional mean square errors with respect to identification and state estimation become minimal separately.

First, the first term of (5.21) is considered in this section. We shall write the conditional probability and the conditional probability density of the event $A^{(i)}$ conditioned by $\mathcal{Y}_t$ by $P(A^{(i)}|\mathcal{Y}_t)$ and $p(A^{(i)}|\mathcal{Y}_t)$ respectively. From (5.6), it is apparent that

$$\hat{\theta}(t) \triangleq E\{\theta|\mathcal{Y}_t\} = \sum_{i=1}^{K} \theta_i P(A^{(i)}|\mathcal{Y}_t). \tag{5.22}$$

The *a posteriori* probability $P(A^{(i)}|\mathcal{Y}_t)$ required in (5.22) can be evaluated by

$$(5.23)\qquad P(A^{(i)}|\mathcal{Y}_t) = \frac{p(\mathcal{Y}_t|A^{(i)})P(A^{(i)})}{\sum_{j=1}^{K} p(\mathcal{Y}_t|A^{(j)})P(A^{(j)})},$$

where $P(A^{(i)})$ is the *a priori* probability in (5.6) and $P(A^{(i)})=P(\theta_i)$. From (5.23), it is a simple exercise to show that

$$(5.24)\qquad P(A^{(i)}|\mathcal{Y}_t) = [\sum_{j=1}^{K} \alpha_{ji}\Lambda_{ji}(t)]^{-1} = M_i(t)$$

where $\Lambda_{ji}(t)$ is the modified likelihood-ratio function defined by

$$(5.25)\qquad \Lambda_{ji}(t) = \frac{p(\mathcal{Y}_t|A^{(j)})}{p(\mathcal{Y}_t|A^{(i)})}, \qquad (i,j=1,2,\cdots,K)$$

and

$$(5.26)\qquad \alpha_{ji} = \frac{P(A^{(j)})}{P(A^{(i)})}.$$

Hence, the optimal estimation $\hat{\theta}(t)$ given by (5.22) becomes

$$(5.27)\qquad \hat{\theta}(t) = \sum_{i=1}^{K} \theta_i M_i(t).$$

In order to compute recursively the optimal estimate $\hat{\theta}(t)$ in the form of

$$(5.28)\qquad d\hat{\theta}(t) = \sum_{i=1}^{K} \theta_i dM_i(t),$$

the following two theorems are stated.

Theorem 5.1. The modified likelihood-ratio function $\Lambda_{ji}(t)$ defined by (5.25) is determined by

$$(5.29)\qquad d\Lambda_{ji}(t) = \Lambda_{ji}(t)\{\hat{h}_j(t,u_t)-\hat{h}_i(t,u_t)\}R^{-2}(t)\{dy(t)-\hat{h}_i(t,u_t)dt\}$$

or

$$(5.30)\qquad d[\ln\Lambda_{ji}(t)] = \frac{1}{2}[\hat{h}_j(t,u_t)-\hat{h}_i(t,u_t)]R^{-2}(t)$$

$$\times\{2dy(t)-[\hat{h}_j(t,u_t)+\hat{h}_i(t,u_t)]dt\}$$

with the initial condition,

$$(5.31)\qquad \Lambda_{ji}(0) = 1, \quad \text{for } i,j=1,2,\cdots,K.$$

Proof. Noting that, from (5.18) and (5.25)

$$(5.32)\qquad \Lambda_{ji}(t) = \frac{\Lambda_j(t)}{\Lambda_i(t)}$$

and using Lemma 5.3, we have

$$(5.33)\qquad \Lambda_{ji}(t) = \exp\{\int_0^t[\hat{h}_j(s,u_s)-\hat{h}_i(s,u_s)]R^{-2}(s)dy(s)$$

$$-\frac{1}{2}\int_0^t[\hat{h}_j^{\,2}(s,u_s)-\hat{h}_i^{\,2}(s,u_s)]R^{-2}(s)ds\}.$$

Hence

$$(5.34)\qquad d\Lambda_{ji}(t) = \Lambda_{ji}(t)[\exp\{(\hat{h}_j-\hat{h}_i)R^{-2}(t)dy(t)$$

$$-\frac{1}{2}(\hat{h}_j^{\,2}-\hat{h}_i^{\,2})R^{-2}(t)dt\} - 1].$$

Expanding the exponential function in (5.34) and deleting the terms of a higher order than $(dt)^{3/2}$, the final result can be obtained.

Theorem 5.2. The sample process of the $M_i(t)$-process defined by (5.24) is determined by

$$(5.35)\qquad dM_i(t) = -\sum_{j=1}^{K}\alpha_{ji}\Lambda_{ji}(t)M_i^{\,2}(t)\{\hat{h}_j(t,u_t)-\hat{h}_i(t,u_t)\}R^{-2}(t)$$

$$\times\{dy(t)-\hat{h}_i(t,u_t)dt\} + \sum_{j=1}^{K}\sum_{k=1}^{K}\alpha_{ji}\alpha_{ki}\Lambda_{ji}(t)\Lambda_{ki}(t)M_i^{\,3}(t)$$

$$\times[\hat{h}_j(t,u_t)-\hat{h}_i(t,u_t)][\hat{h}_k(t,u_t)-\hat{h}_i(t,u_t)]R^{-2}(t)dt,$$

where $i=1,2,\cdots,K$.

Theorem 5.2 can be proved via somewhat tedious calculations in the framework of Itô stochastic calculus. A detailed aspect of the proof will be shown in Appendix C.

From (5.27), the covariance of the unknown parameter θ becomes

$$\text{cov.}[\theta|\mathcal{Y}_t] = \sum_{i=1}^{K} \theta_i \theta_i{}' M_i(t) - [\sum_{i=1}^{K} \theta_i M_i(t)][\sum_{i=1}^{K} \theta_i{}' M_i(t)]. \tag{5.36}$$

As described in this section the recursive computation can be performed by (5.28) and (5.35). However, it may be observed by inspection of (5.13) and (5.35) that the running value of the optimal estimate $\hat{u}_i$ is required.

5.4. State Estimation

The optimal estimate $\hat{u}$ is generated by the familiar conditional mean estimator

$$\hat{u}(t,x) = \int_{-\infty}^{\infty} up(t,x,u|\mathcal{Y}_t)du. \tag{5.37}$$

Bearing the assumption (5.7) in mind, the conditional probability density in (5.37) yields

$$p(t,x,u|\mathcal{Y}_t) = \sum_{i=1}^{K} p(t,x,u|\mathcal{Y}_t, A^{(i)})P(A^{(i)}|\mathcal{Y}_t). \tag{5.38}$$

Hence, the optimal estimate defined by (5.37) is

$$\begin{aligned} \hat{u}(t,x) &= \sum_{i=1}^{K} P(A^{(i)}|\mathcal{Y}_t)\int_{-\infty}^{\infty} up(t,x,u|\mathcal{Y}_t, A^{(i)})du \\ &= \sum_{i=1}^{K} M_i(t)\hat{u}_i(t,x), \end{aligned} \tag{5.39}$$

where use of (5.14) and (5.24) have been made. The i-th optimal estimate can be recursively computed by (5.15).

The covariance is defined by

$$S(t,x,z) = E\{[u(t,x)-\hat{u}(t,x)][u(t,z)-\hat{u}(t,z)]|\mathcal{Y}_t\}. \tag{5.40}$$

Since the covariance (5.40) can be written as

$$(5.41) \qquad S(t,x,z) = \sum_{i=1}^{K} E\{u(t,x)u(t,z)|\mathcal{Y}_t, A^{(i)}\}P(A^{(i)}|\mathcal{Y}_t) - \hat{u}(t,x)\hat{u}(t,z),$$

and

$$(5.42) \qquad E\{u(t,x)u(t,z)|\mathcal{Y}_t, A^{(i)}\} = S_i(t,x,z) + \hat{u}_i(t,x)\hat{u}_i(t,z),$$

it follows from (5.24) that

$$(5.43) \qquad S(t,x,z) = \sum_{i=1}^{K} [S_i(t,x,z)+\hat{u}_i(t,x)\hat{u}_i(t,z)]M_i(t) - \hat{u}(t,x)\hat{u}(t,z).$$

An entire aspect of the optimal estimate is performed by use of (5.15), (5.17), (5.29), (5.35), (5.39) and (5.43). Their preassigned initial conditions are $E\{u_i(0,x)\}=\hat{u}_i(0,x)$ for (5.15), $S_i(0,\xi,x)$ for (5.17), (5.31) for (5.29), $M_i(0)=P(A^{(i)})$ for (5.35), $\hat{u}(0,x)=\Sigma_{i=1}^{K}M_i(0)\cdot\hat{u}_i(0,x)$ for (5.39) and $S(0,x,z)=\Sigma_{i=1}^{K}[S_i(0,x,z)+\hat{u}_i(0,x)\hat{u}_i(0,z)]M_i(0) -\hat{u}(0,x)\hat{u}(0,z)$ for (5.43). The coupled identification-estimation procedure proposed here is schematically illustrated by Fig.5.1.

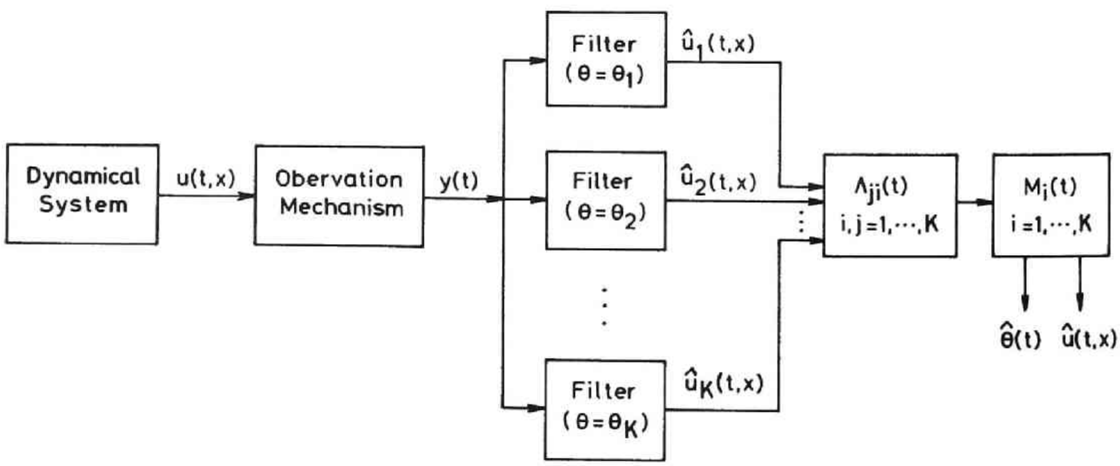

Fig.5.1. Schematic diagram for calculating the estimates of parameter θ and the state u(t,x).

Remark 5.3: An extension to the case $N\geq 2$. For instance, we shall consider the case where $N=2$. In this case, instead of (5.7), the following $K\times K$ joint events should be considered, i.e.

$$A^{(ij)} = \{\omega:\ \theta^{(1)}(\omega)=\theta_i,\ \theta^{(2)}(\omega)=\theta_j\} \quad (i,j=1,2,\cdots,K).$$

Thus, although the theoretical approach is still applicable, the recursive computation becomes considerably complicated.

5.5. Numerical Examples

5.5.1. *Example-5.1.*

The one-dimensional distributed parameter system is considered. For $x\varepsilon[0,1]$, $t\varepsilon[0,T]$, the mathematical model is given by

$$\text{(5.44)} \qquad du(t,x) = [B\frac{\partial^2}{\partial x^2}u(t,x)]dt + C\delta(x-\theta)dt + Gdw(t,x)$$

with the associated initial and boundary conditions,

$$\text{(5.45a)} \qquad u(0,x) = A\sin^2 \pi x, \quad x\varepsilon[0,1],$$

$$\text{(5.45b)} \qquad u(t,x) = 0 \quad \text{at } x=0,\ 1,$$

where A, B, C and G are all constants.

The observation mechanism is

$$\text{(5.46a)} \qquad dy(t) = [\int_0^1 H\delta(z-\eta)u(t,z)dz]dt + Rdv,$$

$$\text{(5.46b)} \qquad y(0) = 0,$$

where both H and R are respectively constants and η shows the location of the measurements.

By using (5.15) and (5.17), dynamics of the state estimator and the associated covariance are determined as

$$\text{(5.47)} \qquad d\hat{u}_i(t,x) = [B\frac{\partial^2}{\partial x^2}\hat{u}_i(t,x)]dt + C\delta(x-\theta_i)dt$$

$$+ HS_i(t,\eta,x)R^{-2}\{dy-H\hat{u}_i(t,\eta)dt\},$$

$$\text{(5.48)} \qquad \frac{\partial}{\partial t}S_i(t,x,z) = B(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2})S_i(t,x,z) + G^2\delta(x-z) -$$

$$- H^2R^{-2}S_i(t,\eta,x)S_i(t,\eta,z).$$

Futhermore, the modified likelihood-ratio function (5.29) can be obtained as

$$(5.49) \qquad d\Lambda_{ji}(t) = \Lambda_{ji}(t)H\{\hat{u}_j(t,\eta)-\hat{u}_i(t,\eta)\}R^{-2}\{dy(t)-H\hat{u}_i(t,\eta)dt\}$$

with $\Lambda_{ji}(0)=1$.

For convenience of the simulation experiment, we assume that the *a priori* probability of the event A_i is uniformly distributed, i.e. $P(A^{(i)})=P(A^{(j)})$. This implies from (5.26) that $\alpha_{ji}=1$ for all i, j. According to this assumption, the definition (5.24) is simply expressed by

$$(5.50) \qquad M_i(t) = [\sum_{j=1}^{K} \Lambda_{ji}(t)]^{-1}.$$

The optimal estimate $\hat{\theta}$ of the unknown parameter can thus be computed by combining (5.27) with (5.50) or by (5.28) and

$$(5.51) \qquad dM_i(t) = -\sum_{j=1}^{K}\Lambda_{ji}(t)M_i^{\,2}(t)H\{\hat{u}_j(t,\eta)-\hat{u}_i(t,\eta)\}R^{-2} \times\{dy(t)-H\hat{u}_i(t,\eta)dt\}$$
$$+\sum_{j=1}^{K}\sum_{k=1}^{K}\Lambda_{ji}(t)\Lambda_{ki}(t)M_i^{\,3}(t)H^2\{\hat{u}_j(t,\eta)-\hat{u}_i(t,\eta)\} \times\{\hat{u}_k(t,\eta)-\hat{u}_i(t,\eta)\}R^{-2}dt.$$

The problem is simulated on a high speed digital computer. The computing procedure is stated in the following steps:

(i) *Write the partial differential equation as the mathematical model of the system with associated initial and boundary conditions.* In (5.44) and (5.45), the values of known parameters were A=1.0, B=1.0, C=500 and G=0.45 respectively.

(ii) *Determine measurement locations in the spatial domain.* The mathematical model (5.46) implies that measurement at a preassigned location η is currently made with respect to time, where two trials

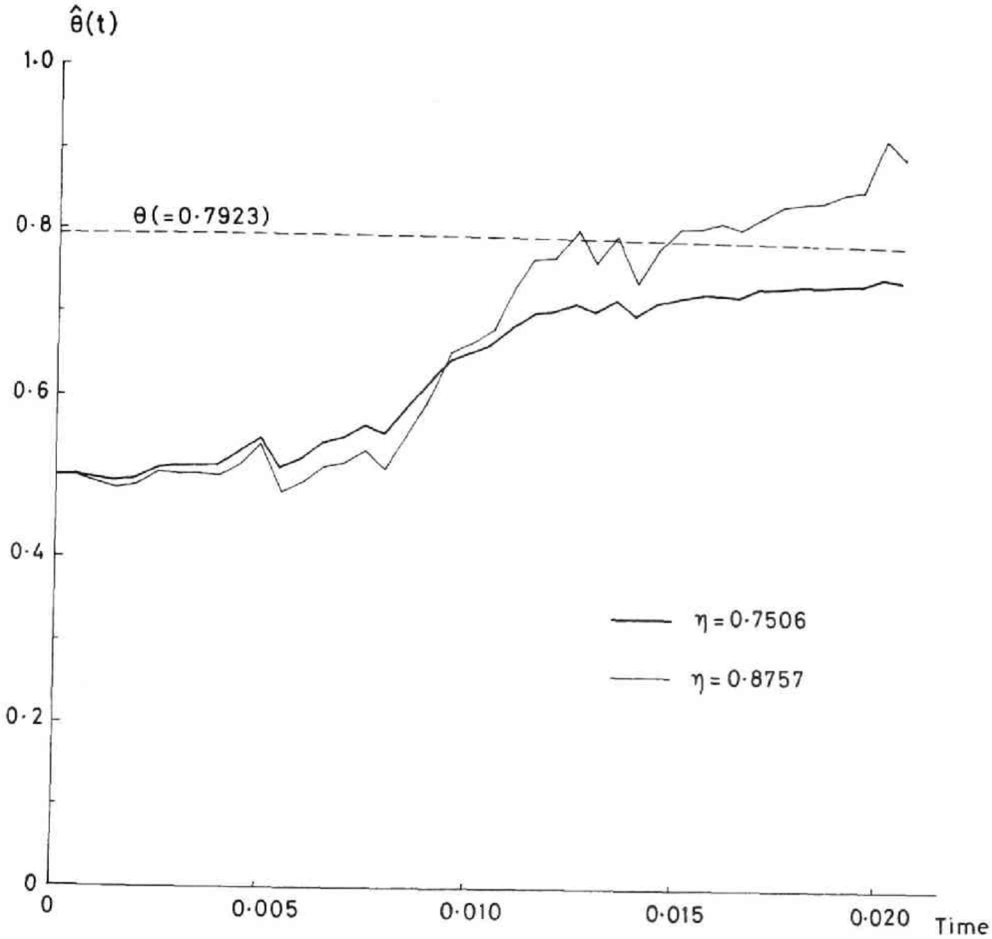

Fig.5.2. The $\hat{\theta}(t)$-runs in Example-5.1.

were made on the choice of the measurement locations, i.e. η=0.7506 and η=0.8757 with the same values of H=4.0 and R=0.2, and where we assumed that $Q(x,z)=\delta(x-z)$.

(iii) *Preassign the number of numerical classes* M *of unknown parameter* θ. Investigators are free to choose the number of numerical classes of unknown parameter θ. The particular choice depends on the situation of the problems which are being considered. A choice that M=7 was given in the simulation experiments and θ_i was taken as $\theta_i=i/8$, where $i=1,2,\cdots,7$.

(iv) *Compute sample runs of state estimate, the associated covariance*

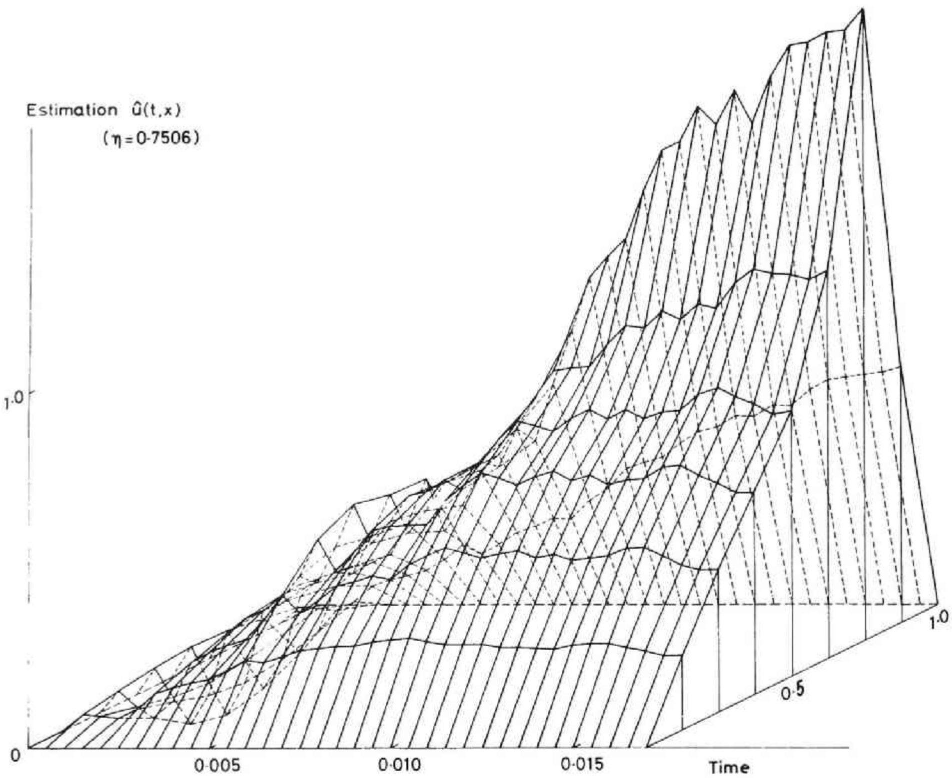

Fig.5.3. The $\hat{u}(t,x)$-run in Example-5.1.

and the likelihood-ratio function. In this example, the initial values of (5.47) and (5.48) were respectively set as $\hat{u}_i(0,x)=0$ and $S_i(0,x,z)=\sin^2\pi x \sin^2\pi z$. Sample runs were obtained by simulating both (5.47) and (5.48) simultaneously on a digital computer with the partitions $\Delta x=1/24$ and $\Delta t=0.0005$ in the spatial variable and in time. By using the run of the state estimate, a sample run of the $\Lambda_{ji}(t)$-process was also computed simultaneously by (5.49) with the initial condition $\Lambda_{ji}(0)=1$.

A sample run of $M_i(t)$ given by (5.50) was applied to both (5.27) and (5.39). Figure 5.2 shows two sample runs of the $\hat{\theta}(t)$-process with

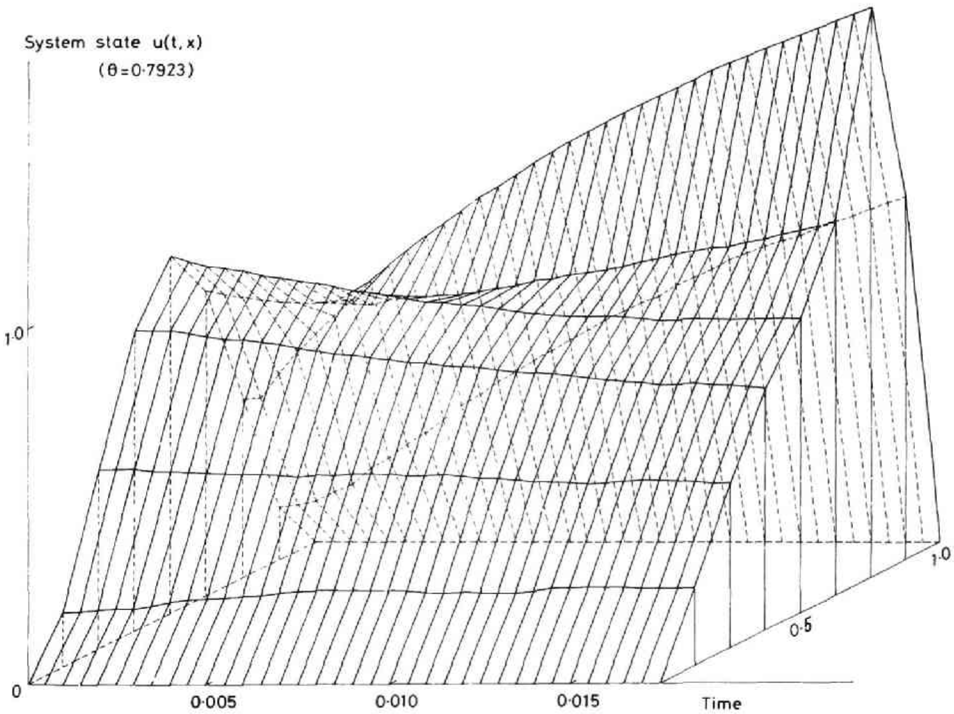

Fig.5.4. The u(t,x)-run representing the system state.

η=0.7506 and 0.8757 respectively. One may understand that the nearer measurement location to the true value θ=0.7923 shows the better identification process $\hat{\theta}(t)$. Figure 5.3 depicts the $\hat{u}(t,x)$ run with η=0.7506. For the purpose of comparative inspection, a sample run of the system state determined by (5.44) was obtained with the associated initial and boundary conditions $u(0,x)=\sin^2\pi x$ and $u(t,0)=u(t,1)=0$ as shown in Fig.5.4. Figure 5.5 shows the u(t,x) and $\hat{u}(t,x)$ runs at the spatial locations of x=0.5 and x=0.75.

Although it is extremely difficult to examine the convergence problem of the filter from theoretical point of view, one way is to observe

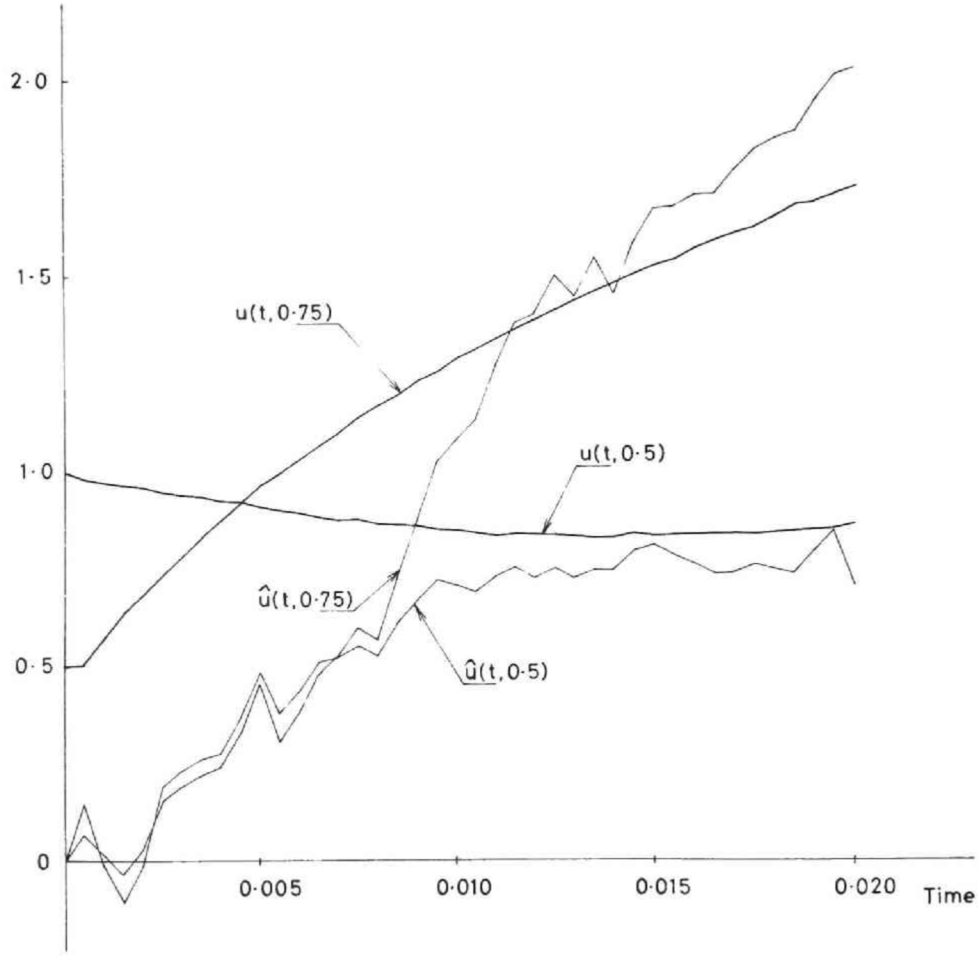

Fig.5.5. The $\hat{u}(t,x)$ and $u(t,x)$ runs at the spatial points $x=0.5$ and $x=0.75$ in Example-5.1.

sample runs of the error covariance $S(t,x,x)$ as illustrated in Fig.5.6.

5.5.2. *Example-5.2.*

Another simulation experiment was performed by adopting a somewhat different observation model from that in Example-5.1. The observation mechanism was set as

$$dy(t) = [\int_0^1 HY(z,\tfrac{1}{2})u(t,z)dz]dt + Rdv, \tag{5.52}$$

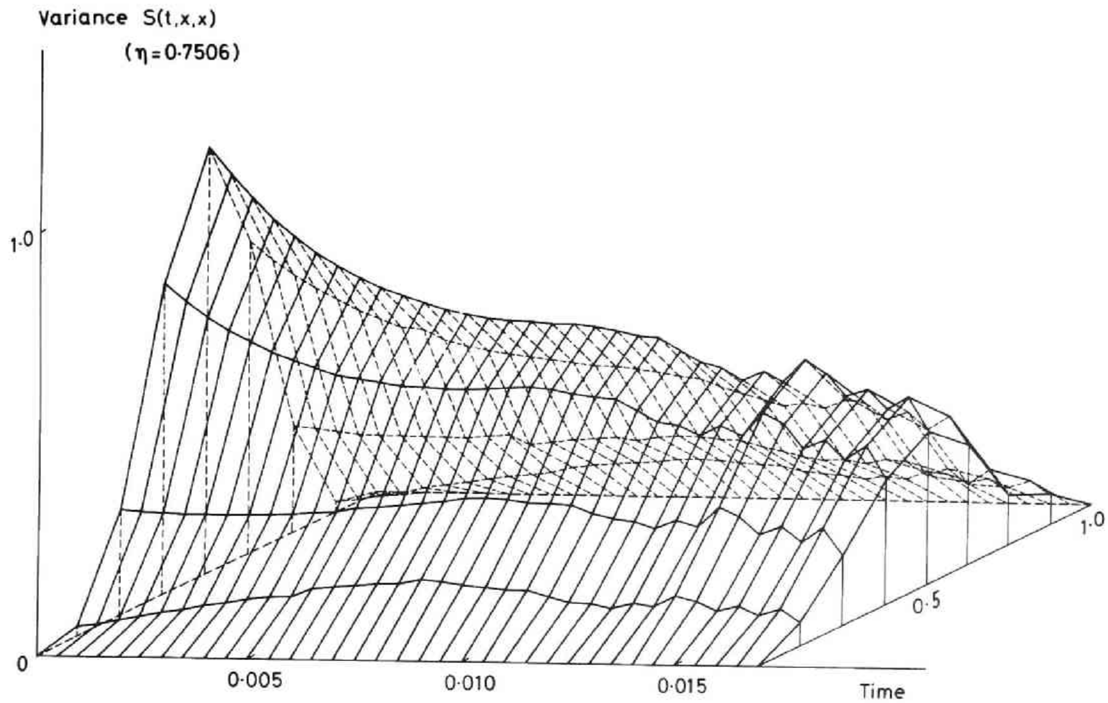

Fig.5.6. The S(t,x,x)-run in Example-5.1.

where the system dynamics was the same as (5.44) with (5.45) and $Y(z,\cdot)$ is the Heaviside's step function, i.e.

$$(5.53) \qquad Y(z,\tfrac{1}{2}) = \begin{cases} 0 & \text{for } z<\frac{1}{2} \\ 1 & \text{for } z\geq\frac{1}{2}. \end{cases}$$

With (5.53), the mathematical model (5.52) is written in a simplified form

$$(5.54) \qquad dy(t) = [\int_{0.5}^{1} Hu(t,z)dz]dt + Rdv.$$

The equations corresponding to (5.47), (5.48), (5.49), (5.51) are respectively given by the following,

$$(5.55) \qquad d\hat{u}_i(t,x) = [B\frac{\partial^2}{\partial x^2}\hat{u}_i(t,x)]dt + C\delta(x-\theta_i)dt$$
$$+ [\int_{0.5}^{1} HS_i(t,x,z)dz]R^{-2}\{dy(t)-[\int_{0.5}^{1} H\hat{u}_i(t,z)dz]dt\},$$

$$(5.56)\qquad \frac{\partial}{\partial t}S_i(t,x,z) = B(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2})S_i(t,x,z) + G^2\delta(x-z)$$

$$- [\int_{0.5}^{1} HS_i(t,\xi,x)d\xi]R^{-2}[\int_{0.5}^{1} HS_i(t,\xi,z)d\xi],$$

$$(5.57)\qquad d\Lambda_{ji}(t) = \Lambda_{ji}(t)[\int_{0.5}^{1} H\{\hat{u}_j(t,z)-\hat{u}_i(t,z)\}dz]R^{-2}$$

$$\times\{dy(t)-[\int_{0.5}^{1} H\hat{u}_i(t,z)dz]dt\},$$

and

$$(5.58)\qquad dM_i(t) = -\sum_{j=1}^{K}\Lambda_{ji}(t)M_i^2(t)[\int_{0.5}^{1} H\{\hat{u}_j(t,z)-\hat{u}_i(t,z)\}dz]R^{-2}$$

$$\times\{dy(t)-[\int_{0.5}^{1} H\hat{u}_i(t,z)dz]dt\}$$

$$+\sum_{j=1}^{K}\sum_{k=1}^{K}\Lambda_{ji}(t)\Lambda_{ki}(t)M_i^3(t)[\int_{0.5}^{1} H\{\hat{u}_j(t,z)-\hat{u}_i(t,z)\}dz]$$

$$\times[\int_{0.5}^{1} H\{\hat{u}_k(t,z)-\hat{u}_i(t,z)\}dz]R^{-2}dt.$$

A variety of single runs was also simulated for Example-5.2. The results presented below are representative of the simulation experiments. In all experiments, the computer program for the simulation follows that for Example-5.1 with the same values of parameters as described previously. Figures 5.7 and 5.8 are respectively the $\hat{\theta}(t)$ and $\hat{u}(t,x)$ runs. Figure 5.9 shows the convergence feature of the $S(t,x,x)$ run.

On the basis of Fig.5.2 to Fig.5.9, as well as on the basis of many other runs not presented here, it is seen that both parameter identification and state estimation depend simultaneously and strongly on the dynamics of the observation mechanism adopted. From the viewpoint of the related covariance to the state estimate, the observation dynamics in Example-5.2 might be more pleasant than in Example-5.1, because in Example-5.2 the observation data is more widely collected than in Example-5.1. However, for the parameter identification runs $\hat{\theta}(t)$ as

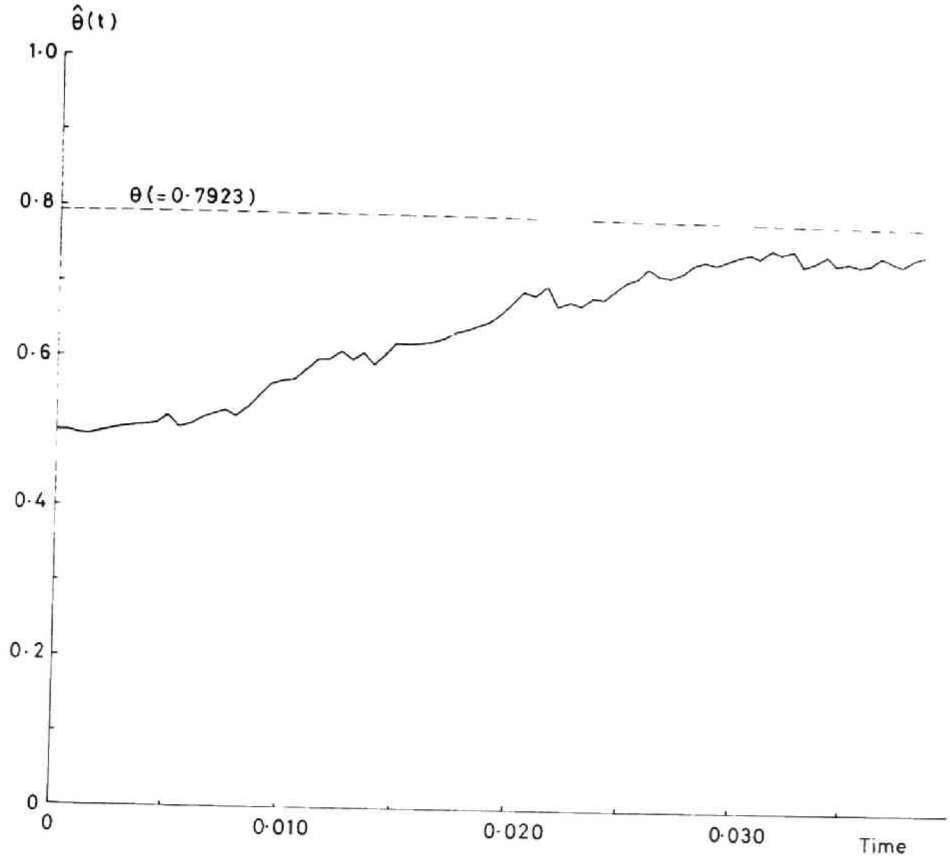

Fig.5.7. The $\hat{\theta}(t)$-run in Example-5.2.

shown in Fig.5.2 and 5.7, the difference between the observation dynamics of (5.46) and (5.52) is not so remarkable because the identification scheme in Example-5.1 depends on η which indicates the spatial location of an observer. The final point to be discussed is thus related to the problem of finding the optimal form of observation dynamics. This is somewhat a difficult problem because, at the present time, there is no simple and systematic procedure with a mathematical background. Trials proposed in this paper on various dynamics of the observation mechanisms are one way to see the feasibility of the solution of the simultaneous aspect of parameter identification and state estimation for distributed

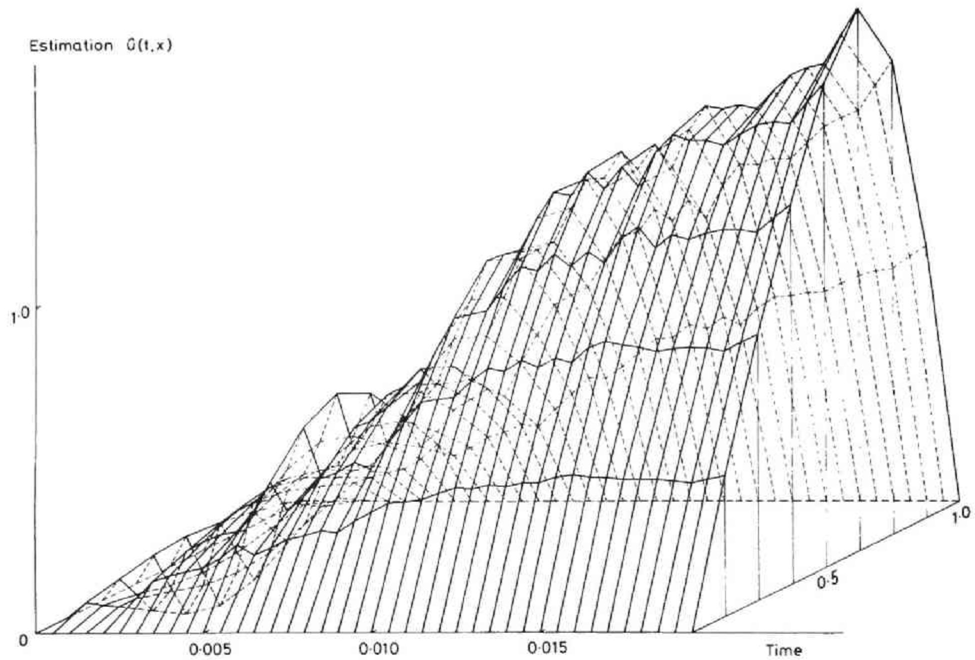

Fig.5.8. The $\hat{u}(t,x)$-run in Example-5.2.

systems.

5.6. Discussions and Summary

A method has been presented for the identification of unknown constant parameters and state estimation in distributed systems which can be modeled by partial differential equations with the specified initial and boundary conditions.

The basic notion of the method developed here is the separation principle of the identification scheme from the state estimation. With this concept, a saving in computation time and computer storage requirements is achieved in comparison with familiar methods in which the system

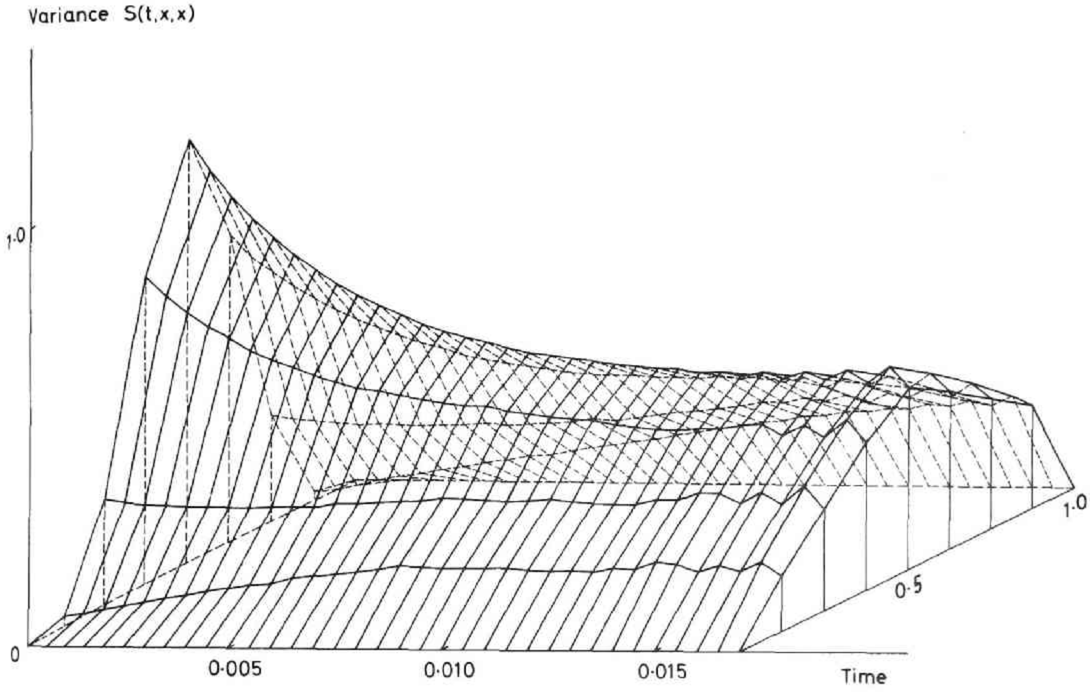

Fig.5.9. The S(t,x,x)-run in Example-5.2.

state and unknown parameter vectors combine and form a new state vector. The major saving in the computational scheme in this chapter is that there is no need to compute the covariance function between the system state and unknown parameters. It is not one of the purposes of this chapter to compare the proposed method with different identification schemes.

The requirement in this chapter is to show that the parameter identification algorithm for a partial differential equation is performed by using the Bayesian approach and the filtering technique in the Markovian framework.

CHAPTER 6. OPTIMAL STOCHASTIC CONTROL FOR NONLINEAR DISTRIBUTED PARAMETER SYSTEMS WITH COMPLETE STATE INFORMATION

6.1. Introductory Remarks

Practical examples of the optimal control problem are found in the control of temperature profiles in a catalytic reactor or a furnace, the control of diffusions due to random excitation in environmental systems the control of reactions in the chemical plants, the control for the prevention of air pollution in urban systems, etc.

For the linear and/or nonlinear systems, significant advances in stochastic control problems were made by several investigators, as surveyed in Subsec.1.1.C, Sec.1.1, Chap.1.

It is well known that dynamical systems to be controlled exhibit various kinds of nonlinear characteristics, and also that the optimal control problem of such nonlinear distributed systems has received considerable attentions in recent years. Up to the present time, a number of studies concerning the so-called Linear-Quadratic-Gaussian (LQG) problem have attained a certain degree of maturity with respect to both theoretical and algorithmic advances, as well as current and potential future applications (cf. Subsec.1.1.C, Sec.1.1, Chap.1 in Part One). On the other hand, the nonlinear problem contains inherent difficulties in itself. A ray of hope to solve such a problem will be only approximations for nonlinear functions to certain linear ones. The author, in this chapter, proposes an approximate method of stochastic optimal control for a class of D.P.S. which is described by stochastic nonlinear partial differential equation, along the line of the LQG context, extending the stochastic linearization technique presented in Sec.3.3, Chap.3.

6.2. Problem Statements

In this chapter, we are concerned with a control problem of nonlinear D.P.S. under the complete state information. The mathematical model considered is Σ_3 defined in Def.2.3, Sec.2.4, i.e.

$$(6.1)\qquad du(t,x) = F(t,x,u,u_x,u_{xx})dt + C(t,x)f(t,x)dt + G(t,x,u)dw(t,x) \quad : \quad \Sigma_3$$

$$\text{I.C.}\quad u(0,x) = \phi(x), \quad x\varepsilon D$$

$$\text{B.C.}\quad u(t,x) = 0, \quad x\varepsilon\partial D.$$

In the sequel, we shall assume that the system (6.1) with its initial-boundary conditions is *bien posê* in the sense of Hadamard; i.e. the solution of (6.1) uniquely exists and depends continuously on the initial and boundary data.

The problem is to find a control function f so as to minimize the scalar functional,

$$(6.2)\qquad J(f) = E\{\int_0^T[\int_D\int_D M(s,x,z)u(s,x)u(s,z)dzdx + \int_D N(s,x)f^2(s,x)dx]ds\},$$

based on the *a priori* probability distribution of the initial condition $\phi(\cdot)$, where M and N are respectively symmetric (in x and z), nonnegative on D×D and positive on D.

Let f(t,x) be a process such that, for each $t\varepsilon[0,T]$ and $x\varepsilon D$, f(t,x) satisfies the condition,

$$\text{(C6.1)} \qquad \int_0^T E\{|f(t,x)|^2\}dt < \infty.$$

Let $\Psi([0,T]\times D)$ be the class of the $f(t,x)$-process which satisfies (C6.1) and does not violate the existence and uniqueness of the solution of (6.1) (w.p.1) and which tends uniformly to zero as $x\to\partial D$ for all $t\in[0,T]$. The control function $f(t,x)$ is said to be admissible if $f(t,x)$ is the element of $\Psi([0,T]\times D)$. In the sequel, the class of admissible controls is simply expressed by Ψ.

6.3. Basic Hamilton-Jacobi-Bellman Equation

The optimal control problem will be solved by using the method of dynamic programming[15,153].

For (6.2), define a minimal cost functional,

$$\text{(6.3)} \qquad V(t,\kappa) \triangleq \min_{f\in\Psi} E_\kappa\{\int_t^T[\int_{D\times D} M(s,x,z)u(s,x)u(s,z)dzdx + \int_D N(s,x)f^2(s,x)dx]ds\},$$

where $\kappa(x)=u(t,x)$ at time $t\in[0,T]$, and $E_\kappa\{\cdot\}$ denotes the conditional expectation conditioned by $\kappa(x)$. Applying the principle of optimality to the cost functional and using the functional Taylor series expansion [152], the following partial integro-differential equation is obtained:

$$\text{(6.4)} \qquad -\frac{\partial V(t,\kappa)}{\partial t} = \min_{f\in\Psi}[\int_D\{\int_D M(t,x,z)\kappa(x)\kappa(z)dz + \frac{\partial V(t,\kappa)}{\partial\kappa(x)}[F(t,x,\kappa,\kappa_x,\kappa_{xx})+C(t,x)f(t,x)] + N(t,x)f^2(t,x)\}dx + \frac{1}{2}\int_{D\times D}\frac{\delta^2 V(t,\kappa)}{\delta\kappa(x)\delta\kappa(z)}G(t,x,\kappa(x))Q(x,z)G(t,z,\kappa(z))dzdx].$$

Minimization in the right-hand side of (6.4) with respect to f gives the optimal control,

$$\text{(6.5)} \qquad f^o(t,x) = -\frac{1}{2}N^{-1}(t,x)C(t,x)\frac{\delta V(t,\kappa)}{\delta\kappa(x)}.$$

Substituting (6.5) into (6.4), we have the following basic Hamilton-Jacobi-Bellman equation,

$$(6.6)\qquad -\frac{\partial V(t,\kappa)}{\partial t} = \int_D [\int_D M(t,x,z)\kappa(x)\kappa(z)dz + \frac{\delta V(t,\kappa)}{\delta\kappa(x)} F(t,x,\kappa,\kappa_x,\kappa_{xx})$$

$$- \frac{1}{4}N^{-1}(t,x)C^2(t,x)(\frac{\delta V(t,\kappa)}{\delta\kappa(x)})^2]dx$$

$$+ \frac{1}{2}\int_{D\times D} \frac{\delta^2 V(t,\kappa)}{\delta\kappa(x)\delta\kappa(z)} G(t,x,\kappa(x))Q(x,z)G(t,z,\kappa(z))dzdx$$

with its terminal condition,

$$(6.7)\qquad V(T,\kappa) = 0.$$

6.4. Suboptimal Control for Nonlinear D.P.S. with State-Independent Noise

In this section, an extended method of stochastic linearization presented in Sec.3.3, Chap.3 is used for deriving the suboptimal control. For a while, we set as $G(t,x,u)=G_0(t,x)$. Define a new $[n(n+1)/2+n+1]$-dimensional vector

$$(6.8)\qquad v = [v_2{}'\ v_1{}'\ v_0]'$$

with components,

$$(6.9)\qquad \begin{cases} v_2 = [u_{11}\ u_{12}\ \cdots\ u_{1n}\ u_{22}\ \cdots\ u_{2n}\ u_{33}\ \cdots\ u_{nn}]' \\ v_1 = [u_1\ u_2\ \cdots\ u_n]' \\ v_0 = u, \end{cases}$$

where $u_i=\partial u/\partial x_i$ and $u_{ij}=\partial^2 u/\partial x_i \partial x_j$ $(i,j=1,2,\cdots,n)$ and the prime denotes the transpose of a vector.

For each $x\varepsilon D$, we expand the nonlinear function $F(t,x;v)\triangleq F(t,x,u,u_x,u_{xx})$ into

$$(6.10)\qquad F(t,x;v) = a(t,x) + B'(t,x)(v-\bar{v}) + e(t,x),$$

where

$$(6.11)\qquad B(t,x) = [b_2{}'(t,x)\ b_1{}'(t,x)\ b_0(t,x)]'$$

with components,

$$
(6.12) \qquad \begin{cases} b_2(t,x) = [(b_2)_{11}\ (b_2)_{12} \cdots (b_2)_{1n}\ (b_2)_{22}\ (b_2)_{23} \cdots (b_2)_{nn}]' \\ b_1(t,x) = [(b_1)_1\ (b_1)_2\ \cdots\ (b_1)_n]' \\ b_0(t,x) = b_0. \end{cases}
$$

In (6.10), the term $e(t,x)$ is the collection of error terms and the symbol "$\bar{\cdot}$" denotes $E\{\cdot|\phi(x)\}$, so that

$$
(6.13) \qquad \bar{v} = [\bar{v}_2'\ \bar{v}_1'\ \bar{v}_0]'.
$$

Both $a(t,x)$ and $B(t,x)$ are the coefficients of the expansion determined by such a way that, for each $x \varepsilon D$, $E\{|F(t,x;v)-[a(t,x)+B'(t,x)(v-\bar{v})]|^2|\phi(x)\}$ becomes minimal with respect to $a(t,x)$ and $B(t,x)$. A simple calculation gives that the necessary and sufficient conditions for $\min_{a,B} E\{|e(t,x)|^2|\phi(x)\}$ for each $x \varepsilon D$ are given by

$$
(6.14) \qquad a(t,x) = E\{F(t,x;v)|\phi(x)\} = \bar{F}(t,x;v)
$$

$$
(6.15) \qquad B(t,x) = S^{-1}(t,x) E\{(v-\bar{v})[F(t,x;v)-\bar{F}(t,x;v)]|\phi(x)\},
$$

where

$$
(6.16) \qquad S(t,x) = E\{(v-\bar{v})(v-\bar{v})'|\phi(x)\}.
$$

By using the above linearization, the nonlinear process (6.1) is replaced by the approximated one,

$$
\begin{aligned}
(6.17) \qquad d\bar{u}(t,x) &\overset{\sim}{=} \{a(t,x)+B'(t,x)(v-\bar{v})\}dt + C(t,x)f(t,x)dt \\
&\qquad + G_0(t,x)dw(t,x) \\
&= \{L_x[u(t,x)-\bar{u}(t,x)]+a(t,x)\}dt + C(t,x)f(t,x)dt \\
&\qquad + G_0(t,x)dw(t,x),
\end{aligned}
$$

where the approximate linear operator $L_x(\cdot)$ is given by

$$
\begin{aligned}
(6.18) \qquad L_x(\cdot) = &\sum_{\substack{i,j=1 \\ i\le j}}^{n} \{b_2(t,x)\}_{ij} \frac{\partial^2}{\partial x_i \partial x_j}(\cdot) + \sum_{i=1}^{n} \{b_1(t,x)\}_i \frac{\partial}{\partial x_i}(\cdot) \\
&+ b_0(t,x)(\cdot).
\end{aligned}
$$

It may easily be shown that the coefficients a and B depend on both $\bar{v}$ and S which are expressed in terms of $\bar{u}$ and P, where $\bar{u}$ and P are respectively defined by

$$(6.19)\qquad \begin{cases} \bar{u} = E\{u|\phi(x)\} \\ P(t,x,z) = E\{[u(t,x)-\bar{u}(t,x)][u(t,z)-\bar{u}(t,z)]|\phi(x)\} \end{cases}$$

and these are the solutions of the following equations,

$$(6.20)\qquad d\bar{u}(t,x)/dt = a(t,x) + C(t,x)\bar{f}(t,x)$$

$$(6.21)\qquad \begin{aligned} \frac{\partial P(t,x,z)}{\partial t} = {} & (L_x+L_z)P(t,x,z) \\ & + C(t,x)E\{[u(t,z)-\bar{u}(t,z)][f(t,x)-\bar{f}(t,x)]|\phi(x)\} \\ & + C(t,z)E\{[u(t,x)-\bar{u}(t,x)][f(t,z)-\bar{f}(t,z)]|\phi(x)\} \\ & + G_0(t,x)Q(x,z)G_0(t,z). \end{aligned}$$

For the approximated process (6.17), the basic equation (6.6) easily yields

$$(6.22)\qquad \begin{aligned} -\frac{\partial V(t,\kappa)}{\partial t} = {} & \int_D[\int_D M(t,x,z)\kappa(x)\kappa(z)dz \\ & + \frac{\delta V(t,\kappa)}{\delta\kappa(x)}\{L_{t,x}[\kappa(x)-\bar{u}(t,x)]+a(t,x)\} \\ & - \frac{1}{4}N^{-1}(t,x)C^2(t,x)(\frac{\delta V(t,\kappa)}{\delta\kappa(x)})^2]dx \\ & + \frac{1}{2}\int_{D\times D}\frac{\delta^2 V(t,\kappa)}{\delta\kappa(x)\delta\kappa(z)}G_0(t,x)Q(x,z)G_0(t,z)dzdx. \end{aligned}$$

If the original process (6.1) is purely linear, then the corresponding basic equation may be solved by the method of separation of variables. However a striking fact arises in solving (6.22); that is, the fact that (6.22) contains the linearization coefficients a and B which are the functions of the current variables $\bar{u}$, $\bar{u}_x$, $\bar{u}_{xx}$ and P(t,x,x) and that such coefficients prevent us to solve (6.22) in the LQG fashon.

In the following, the author uses a feasible approach which is similar to the method used in Sec.6.6(Method II), Chap.6 in Part One. To do this, during the time interval, $t\leq\tau\leq T$, hold the sample values of a(t,x), B(t,x) and $\bar{u}(t,x)$ as constant, i.e. $a(t,x)=\tilde{a}_t(x)$, $B(t,x)=\tilde{B}_t(x)$

and $\bar{u}(t,x)=\tilde{u}_t(x)$ respectively, and write

$$(6.23)\qquad du(\tau,x) = \tilde{\mathcal{L}}_x u(\tau,x)d\tau + \tilde{s}_t(x)d\tau + C(\tau,x)f(\tau,x)d\tau$$
$$+ G_0(\tau,x)dw(\tau,x), \qquad t\le\tau\le T.$$

In (6.23), the operator $\tilde{\mathcal{L}}_x(\cdot)$ and $\tilde{s}_t(x)$ are respectively given by

$$(6.24)\qquad \tilde{\mathcal{L}}_x(\cdot) = \sum_{\substack{i,j=1\\ i\le j}}^{n} \{\tilde{b}_{2t}(x)\}_{ij} \frac{\partial^2}{\partial x_i \partial x_j}(\cdot) + \sum_{i=1}^{n} \{\tilde{b}_{1t}(x)\}_i \frac{\partial}{\partial x_i}(\cdot)$$
$$+ \tilde{b}_{0t}(x)(\cdot)$$

$$(6.25)\qquad \tilde{s}_t(x) = \tilde{a}_t(x) - \tilde{B}_t'(x)\tilde{v}_t,$$

where $\tilde{v}_t=[\tilde{v}_{2t}'(x)\ \tilde{v}_{1t}'(x)\ \tilde{v}_{0t}(x)]'$.

It follows that, at time τ, the basic equation for the process (6.23) becomes

$$(6.26)\qquad -\frac{\partial V(\tau,\kappa)}{\partial\tau} = \int_{D\times D} M(\tau,x,z)\kappa(x)\kappa(z)dzdx$$
$$+ \int_D \frac{\delta V(\tau,\kappa)}{\delta\kappa(x)}\{\tilde{\mathcal{L}}_x\kappa(x)+\tilde{s}_t(x)\}dx$$
$$- \frac{1}{4}\int_D N^{-1}(\tau,x)C^2(\tau,x)\left(\frac{\delta V(\tau,\kappa)}{\delta\kappa(x)}\right)^2 dx$$
$$+ \frac{1}{2}\int_{D\times D} \frac{\delta^2 V(\tau,\kappa)}{\delta\kappa(x)\delta\kappa(z)} G_0(\tau,x)Q(x,z)G_0(\tau,z)dzdx$$

with the terminal condition

$$(6.27)\qquad V(T,\kappa) = 0,$$

where $\kappa(x)=u(\tau,x)$.

In (6.26) and (6,27), assume that there exists a solution of the following form,

$$(6.28)\qquad V(\tau,\kappa) = \int_{D\times D}\Pi(\tau,x,z)\kappa(x)\kappa(z)dzdx + 2\int_D \alpha(\tau,x)\kappa(x)dx$$
$$+ \beta(\tau),$$

where the scalar functions $\Pi(\tau,x,z)$ (symmetric in x and z), $\alpha(\tau,x)$ and $\beta(\tau)$ are determined by the differential equations which will be given later. Noting the relations

$$\frac{\delta V(\tau,\kappa)}{\delta\kappa(x)} = 2\int_D \Pi(\tau,x,z)\kappa(z)dz + 2\alpha(\tau,x) \tag{6.29}$$

and

$$\frac{\delta^2 V(\tau,\kappa)}{\delta\kappa(x)\delta\kappa(z)} = 2\Pi(\tau,x,z) \tag{6.30}$$

and applying (6.28), (6.29) and (6.30) to (6.26), it follows that

$$f^o(\tau,x) = -N^{-1}(\tau,x)C(\tau,x)[\int_D \Pi(\tau,x,z)\kappa(z)dz+\alpha(\tau,x)] \tag{6.31}$$

and that

$$\frac{\partial\Pi(\tau,x,z)}{\partial\tau} + (\mathcal{L}_x^*+\mathcal{L}_z^*)\Pi(\tau,x,z) - \int_D N^{-1}(\tau,\xi)C^2(\tau,\xi)\Pi(\tau,x,\xi)\Pi(\tau,\xi,z)d\xi + M(\tau,x,z) = 0 \tag{6.32}$$

$$\frac{\partial\alpha(\tau,x)}{\partial\tau} + \mathcal{L}_x^*\alpha(\tau,x) - \int_D N^{-1}(\tau,z)C^2(\tau,z)\Pi(\tau,x,z)\alpha(\tau,z)dz + \int_D \Pi(\tau,x,z)\tilde{s}_t(z)dz = 0 \tag{6.33}$$

$$\frac{d\beta(\tau)}{d\tau} - \int_D N^{-1}(\tau,x)C^2(\tau,x)\alpha^2(\tau,x)dx + 2\int_D \alpha(\tau,x)\tilde{s}_t(x)dx + \int_{D\times D}\Pi(\tau,x,z)G_0(\tau,x)Q(x,z)G_0(\tau,z)dzdx = 0 \tag{6.34}$$

with their terminal-boundary conditions

$$\Pi(T,x,z) = 0,\quad \alpha(T,x) = 0 \text{ and } \beta(T) = 0 \quad \text{for all } x,z\varepsilon D \tag{6.35}$$

$$\Pi(\tau,x,z) = 0 \text{ and } \alpha(\tau,x) = 0 \quad \text{on } x,z\varepsilon\partial D, \tag{6.36}$$

where the boundary conditions are given by the definition of the admissible control which uniformly tends to zero as $x\to\partial D$. In (6.32) and (6.33), the operator $\mathcal{L}_x^*$ denotes the formal adjoint operator of $\mathcal{L}_x$.

6.5. Suboptimal Control for Nonlinear D.P.S. with State-Dependent Noise

Even in the case where the system noise is linearly state-dependent, i.e. $G(t,x,u)=G_1(t,x)u(t,x)$, the parallel discussion holds with the solution $\alpha(\tau,x)$ determined by (6.33) and with the solutions $\Pi(\tau,x,z)$ and $\beta(\tau)$ which are respectively determined by

$$(6.37)\qquad \frac{\partial\Pi(\tau,x,z)}{\partial\tau} + (\tilde{L}_x^*+\tilde{L}_z^*)\Pi(\tau,x,z) + \Pi(\tau,x,z)G(\tau,x)Q(x,z)G(\tau,z)$$

$$-\int_D N^{-1}(\tau,\xi)C^2(\tau,\xi)\Pi(\tau,x,\xi)\Pi(\tau,\xi,z)d\xi + M(\tau,x,z) = 0$$

$$(6.38)\qquad \frac{d\beta(\tau)}{d\tau} - \int_D N^{-1}(\tau,x)C^2(\tau,x)\alpha^2(\tau,x)dx$$

$$+ 2\int_D \alpha(\tau,x)\tilde{s}_t(x)dx = 0$$

with the same terminal-boundary conditions as in (6.35) and (6.36). However, it should be noted that the $\Pi(\tau,x,z)$, $\alpha(\tau,x)$ and $\beta(\tau)$ $(t\leq\tau\leq T)$ make sense only at $\tau=t$, because of the substitution of $\tilde{a}_t(x)$, $\tilde{B}_t(x)$ and $\tilde{u}_t(x)$ for $a(\tau,x)$, $B(\tau,x)$ and $\bar{u}(\tau,x)$. Consequently, at time t, the values of $\Pi(\tau,x,z)|_{\tau=t}$, $\alpha(\tau,x)|_{\tau=t}$ and $\beta(\tau)|_{\tau=t}$ may be used to calculate the coefficients of the solution $V(t,\kappa)$ of (6.22) and to generate the suboptimal control,

$$(6.39)\qquad f^o(t,x) = - N^{-1}(t,x)C(t,x)\{\int_D[\Pi(\tau,x,z)]_{\tau=t}\kappa(z)dz+[\alpha(\tau,x)]_{\tau=t}\}.$$

Applying the suboptimal control (6.39) to (6.20) and (6.21), it follows that

$$(6.40)\qquad \frac{d\bar{u}(t,x)}{dt} = a(t,x) - N^{-1}(t,x)C^2(t,x)\{\int_D[\Pi(t,x,z)\bar{u}(t,z)dz$$

$$+\alpha(t,x)\}$$

$$(6.41)\qquad \frac{\partial P(t,x,z)}{\partial t} = (L_x+L_z)P(t,x,z)$$

$$- \{N^{-1}(t,x)C^2(t,x)\int_D\Pi(t,x,\xi)P(t,\xi,z)d\xi$$

$$+N^{-1}(t,z)C^2(t,z)\int_D\Pi(t,\xi,z)P(t,x,\xi)d\xi\}$$

$$+ G_0(t,x)Q(x,z)G_0(t,z).$$

Thus, an approximate overall configuration of the nonlinear distributed control system (6.1) has been established in a form of a feedback system.

6.6. Digital Simulations

We shall consider the nonlinear distributed parameter system described by

$$(6.42a)\qquad du(t,x) = [\frac{\partial^2 u(t,x)}{\partial x^2} + \beta u^2(t,x)]dt + Cf(t,x)dt$$

$$+ Gdw(t,x),$$

where β is a preassigned positive constant which is not so large and C and G are also positive constants. Both the initial- and boundary-conditions are respectively given by

$$(6.42b)\qquad \mathcal{E}\{u(0,x)\} = A\sin^2 \pi x \quad \text{for } 0\leq x\leq 1$$

$$(6.42c)\qquad u(t,x) = 0 \quad \text{for } x=0,1,$$

where A is a positive constant. The variance of the Brownian motion process is given by

$$(6.43)\qquad Q(x,z) = \delta(x-z) \quad \text{for } 0<x,z<1.$$

The problem is to compute the optimal control $f^o(t,x)$ which minimizes the cost functional,

$$(6.44)\qquad J(f) = \mathcal{E}\{\int_0^T[\int_0^1\int_0^1 M(x,z)u(s,x)u(s,z)dzdx$$

$$+ \int_0^1 Nf^2(s,x)dx]ds\},$$

where M is nonnegative and symmetric in x and z and N is a positive constant.

The linearization coefficients (6.14) and (6.15) are, in this case, respectively calculated by

$$(6.45a)\qquad a(t,x) = \frac{\partial^2\bar{u}(t,x)}{\partial x^2} + \beta[P(t,x,x)+\bar{u}^2(t,x)]$$

$$(6.45b)\qquad B(t,x) = [1 \quad 0 \quad 2\beta\bar{u}(t,x)]'.$$

From (6.39), the suboptimal control is given by

$$(6.46)\qquad f^o(t,x) = -N^{-1}C\{\int_0^1[\Pi(\tau,x,z)]_{\tau=t}u(t,z)dz+[\alpha(\tau,x)]_{\tau=t}\},$$

where Π and α are the solutions of differential equations,

$$(6.47)\qquad \frac{\partial \Pi(\tau,x,z)}{\partial \Pi} + [(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2})\Pi(\tau,x,z) + 2\beta\{\tilde{u}_t(x)+\tilde{u}_t(z)\}\Pi(\tau,x,z)]$$

$$- \int_0^1 N^{-1}C^2\Pi(\tau,x,\xi)\Pi(\tau,\xi,z)d\xi + M(x,z) = 0$$

$$(6.48)\qquad \frac{\partial \alpha(\tau,x)}{\partial \tau} + [\frac{\partial^2}{\partial x^2}\alpha(\tau,x) + 2\beta\tilde{u}_t(x)\alpha(\tau,x)]$$

$$- \int_0^1 N^{-1}C^2\Pi(\tau,x,z)\alpha(\tau,z)dz + \int_0^1 \Pi(\tau,x,z)\tilde{s}_t(z)dz = 0$$

with their terminal-boundary conditions,

$$(6.49)\qquad \Pi(T,x,z) = 0 \quad \text{and} \quad \alpha(T,x) = 0 \quad \text{for } 0\leq x,z\leq 1$$

$$(6.50)\qquad \Pi(\tau,x,z) = 0 \quad \text{and} \quad \alpha(\tau,x) = 0 \quad \text{for } x,z=0 \text{ and } 1.$$

Equations (6.42) to (6.50) are simulated on a digital computer with a similar procedure to that mentioned in Sec.4.5, Chap.4 or in [138]. The standard difference operators D_+, D_- and D_0 are also used in this section. Application of the spatial difference scheme to (6.1) gives a set of increments of the state,

$$(6.51)\qquad \delta u_j(x_i) = u(t_{j+1},x_i) - u(t_j,x_i)$$

$$\cong F(t_j,x_i,u(t_j,x_i),D_0u(t_j,x_i),D_+D_-u(t_j,x_i))\delta t_j$$

$$+ Cf(t_j,x_i)\delta t_j + G\delta w_j(x_i) \quad (i=0,1,\cdots,I-1),$$

where the spatial interval [0,1] is divided into I partitions such that $\delta x_i=x_{i+1}-x_i$. The suboptimal control $f^o(t_j,x_i)$ given by (6.39) is approximated by

$$(6.52)\qquad f^o(t_j,x_i) = - N^{-1}C\{\sum_{k=0}^{I-1}\Pi(t_j,x_i,x_k)u(t_j,x_k)\delta x_k+\alpha(t_j,x_i)\}$$

$$(i=0,1,\cdots,I-1),$$

where both $\Pi(t_j,x_i,x_k)$ and $\alpha(t_j,x_i)$ are respectively the discrete versions of the solutions of (6.32) and (6.33).

As shown in Fig.6.1, the computational procedure is thus established

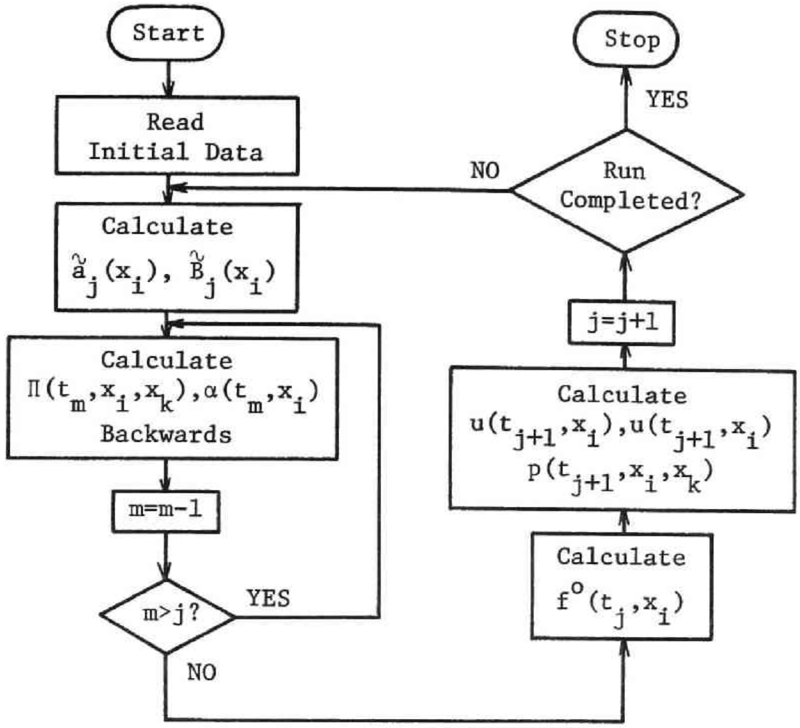

Fig.6.1. Flow diagram of computational procedure.

as the following steps:

(i) Obtain the coefficients $a(t,x)$ and $B(t,x)$ for the preassigned nonlinear function $F(t,x,u,u_x,u_{xx})$ and write their discrete versions, $a(t_j,x_i)$ and $B(t_j,x_i)$.

(ii) Calculate the initial values $\Pi(0,x_i,x_k)$ and $\alpha(0,x_i)$ by solving the partial integro-differential equations for $\Pi(\tau,x,z)$ and $\alpha(\tau,x)$ with their terminal-boundary conditions.

(iii) Determine the initial value of the suboptimal control by

$$f^o(0,x_i) = - N^{-1}C\{\sum_{k=0}^{I-1}\Pi(0,x_i,x_k)\phi(x_k)\delta x_k+\alpha(0,x_i)\}.$$

(iv) By using the values of $\tilde{a}_j(x_i)$, $\tilde{B}_j(x_i)$ and $\tilde{u}_j(x_i)$, compute $\bar{u}(t_{j+1},x_i)$ and $P(t_{j+1},x_i,x_k)$ from (6.40) and (6.41).

(v) Compute $\tilde{a}_{j+1}(x_i)$ and $\tilde{B}_{j+1}(x_i)$ with use of the values $\bar{u}(t_{j+1},x_i)$

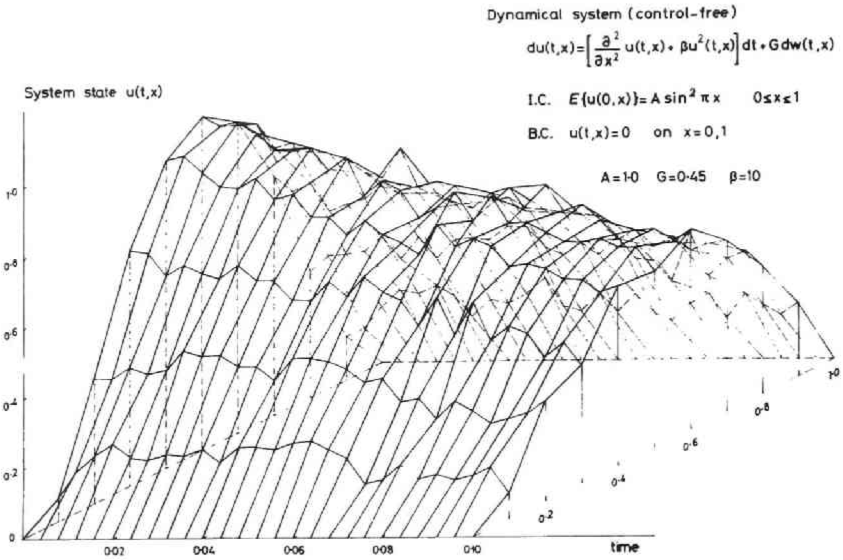

Fig.6.2(a). Control-free with noise.

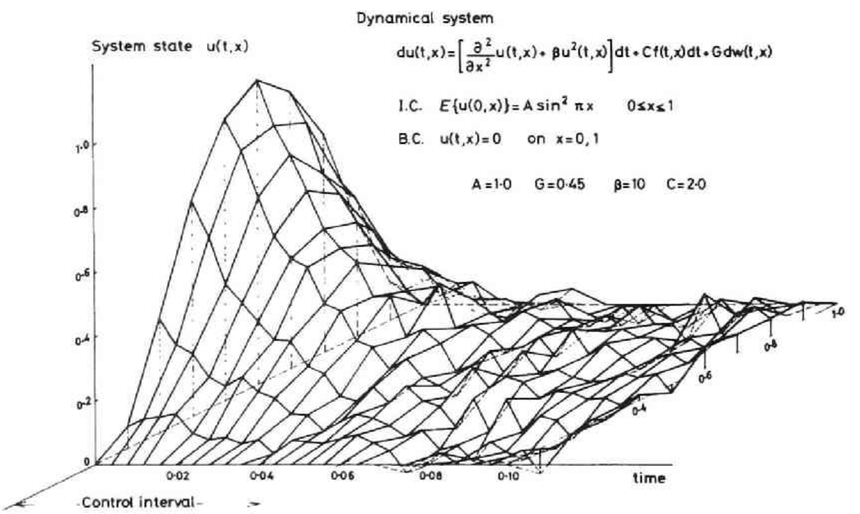

Fig.6.2(b). Controlled with noise.

Fig.6.2. Bird's-eye view of the sample path behavior of the system state u(t,x).

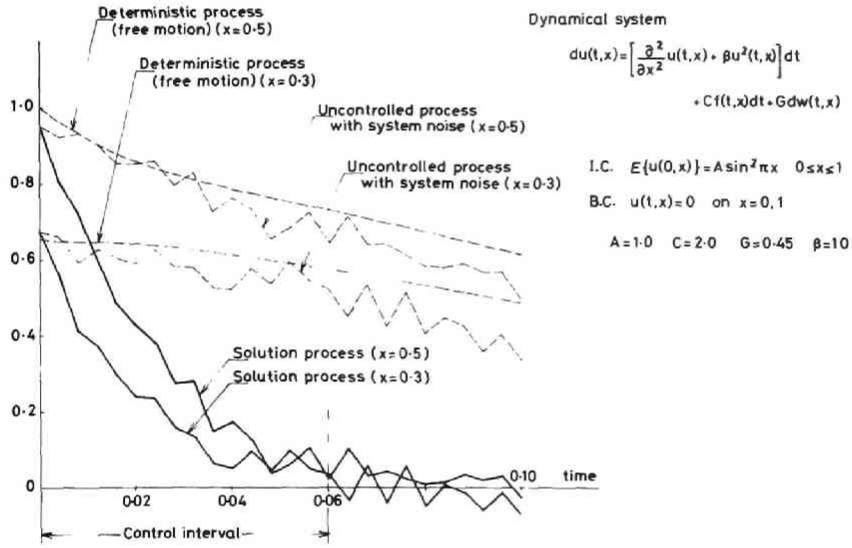

Fig.6.3. Sample paths of the state u(t,x) at x=0.3 and x=0.5.

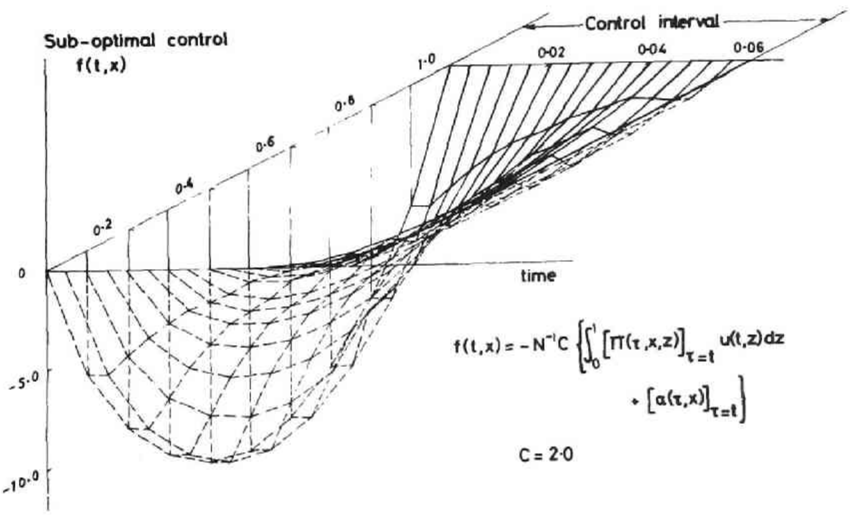

Fig.6.4. Sample path behavior of the suboptimal control f(t,x).

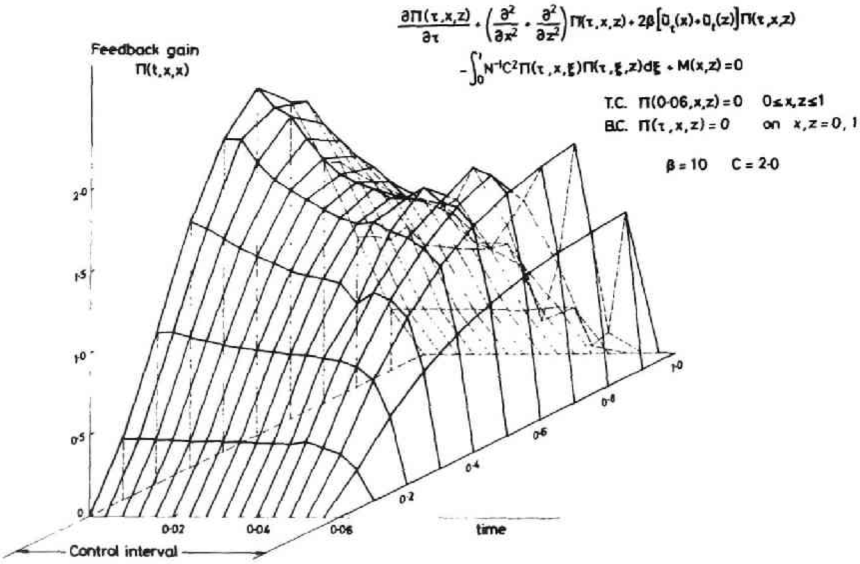

Fig.6.5(a). Run of Π(t,x,x).

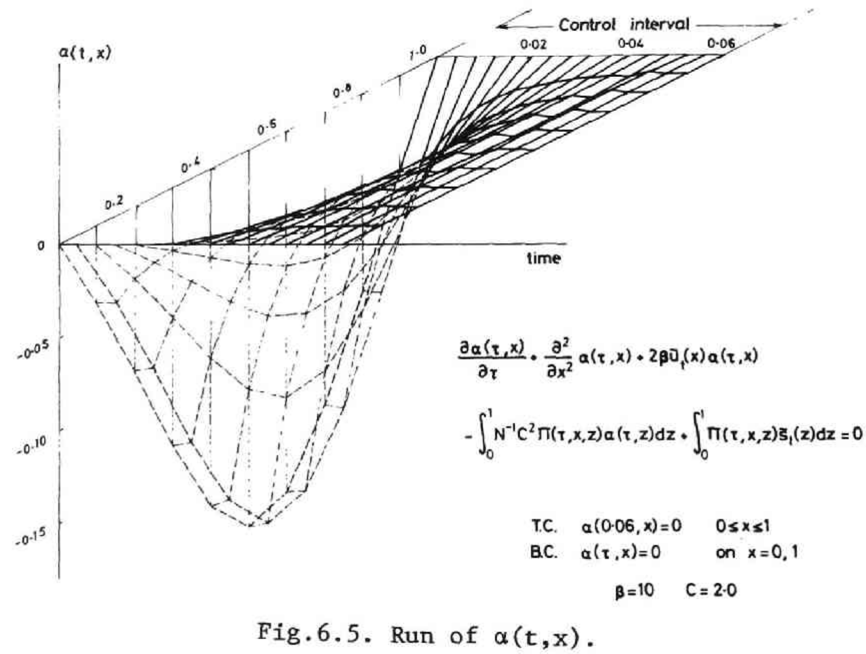

Fig.6.5. Run of α(t,x).

Fig.6.5. Feedback gain Π(t,x,x) and coefficient α(t,x).

and $P(t_{j+1},x_i,x_k)$ determined in Step (iv).

(vi) Obtain $\Pi(t_{j+1},x_i,x_k)$ and $\alpha(t_{j+1},x_i)$ which are determined by (6.32) and (6.33).

(vii) With the newly obtained data $u(t_{j+1},x_i)$ and the values of $\Pi(t_{j+1},x_i,x_k)$ and $\alpha(t_{j+1},x_i)$ obtained in Step (vi), determine the suboptimal control $f^o(t_{j+1},x_i)$ by (6.52).

Letting $j=0,1,\cdots$, the steps (iv) to (vii) give an algorithm to obtain the running values of the suboptimal control $f^o(t_j,x_i)$.

In digital simulations, δx_i and δt_j are given as 0.1 and 0.004 respectively and the control interval is preassigned as [0,0.06]. Figure 6.2 shows the bird's-eye view of the state of the uncontrolled system under system noise and the state of the controlled system described by (6.42), where the coefficients are respectively β=10, C=2.0, G=0.45, A=1, M=50 and N=0.1.

In order to compare the state $u(t,x)$ driven by the suboptimal control $f^o(t,x)$ with the state $u(t,x)$ without control, the convergence of the system states is shown in Fig.6.3 at the spatial locations x=0.3 and x=0.5. From Figs.6.2 and 6.3, it can be seen that an effective role of suboptimal control is recognized at respective locations. Figures 6.4 and 6.5 show the sample paths of the suboptimal control $f(t,x)$ and the associated feedback gain $\Pi(t,x,x)$ and the coefficient $\alpha(t,x)$.

Although it may be extremely difficult to justify analytically the accuracy of the proposed technique, numerical results obtained reveal that the extended stochastic linearization technique developed here is feasible for realizing the stochastic suboptimal control for nonlinear D.P.S.

6.7. Discussions and Summary

In this chapter, via the method of stochastic linearization, a suboptimal control has been obtained for a class of nonlinear D.P.S. with the complete state information. It has been shown that the extended stochastic linearization technique to D.P.S. is attractive for a computer implementation of suboptimal control.

In the procedure to obtain the suboptimal control, the equations of feedback gains Π and α should be solved by the feasible method mentioned in Sec.6.4.

CHAPTER 7. CONCLUSIONS

7.1. Concluding Remarks

In Part Two, some attempts have been made to present a rather general discussion on various aspects of the problems associated with the state estimation, parameter identification and control for nonlinear and/or linear D.P.S., oriented in some parts by the approximation techniques stated in Chap.3. Although some portions of the works may seem to be somewhat abstract from the system engineering point of view, an abstract approach can provide, in general, a better understanding to related problems.

The major difficulties in the computational aspect of distributed parameter control processes are due to the dimensionality of the associated state vectors as pointed out by Bellman[184]. A fresh and effective approach which provides a reduction of dimensionality is certainly required, including computational aspects. In particular, in the problem associated with the nonlinear D.P.S. the curse of dimensionality is the crucial point and this prevents us to perform the operations of estimation, identification and/or control. The proposed methods in Part Two will contribute to obtain feasible solutions to the practical design of D.P.S.

7.2. Discussions

The model of D.P.S. is described by a partial differential equation with additive Gaussian noise, i.e. Eqs.(1.1) or (1.3) in Sec.2.2, Chap.2. However, there are many cases where the coefficients in a system operator are inherently random (cf. Bharucha-Reid[11]). For example, instead of (1.1), a diffusion process in random media is modeled by a partial differential equation,

$$\frac{\partial u(t,x)}{\partial t} = L_x(\omega)u(t,x) + C(t,x)f(t,x), \tag{7.1}$$

where $L_x(\omega)$ is a random (linear) operator. The problems of estimation, identification and control for the system described by (7.1) are the future topics in the theory of distributed parameter control systems. Because of the fact that the theory in this area is not fully developed at the present time, investigations in the immediate future should be directed toward establishing theories to the class of D.P.S. described by (7.1), accompanied with the random eigenvalue problems (cf. Boyce [13]).

APPENDIX A. Proof of Corollary 4.1.

Equation (4.30) is easily derived from (4.25). The version of $\partial P/\partial t$ is evaluated by computing

$$\text{(A.1)} \qquad dP(t,x,z) = d(\widehat{u(t,x)u(t,z)}) - d(\hat{u}(t,x)\hat{u}(t,z)),$$

where "$\hat{\cdot}$" denotes the conditional expectation $E_{(1)}\{\cdot|\mathcal{Y}_t\}$. Let $f(u(t,x), u(t,z))=u(t,x)u(t,z)$ in Theorem 4.1 and use (4.12) to find $d(\widehat{u(t,x)u(t,z)})$. . In the sequel, in order to simplify the notation, if necessary, we shall drop the argument (t,x) and denote the spatial point by superscripts x, z or ξ. Since

$$\text{(A.2)} \qquad \mathcal{G}[u(t,x)u(t,z)] = u^z\mathcal{F}^x + u^x\mathcal{F}^z + G^xG^zQ(x,z),$$

we have

$$\text{(A.3)} \qquad d(\widehat{u^xu^z}) = \widehat{u^z\mathcal{F}^x}dt + \widehat{u^x\mathcal{F}^z}dt + \widehat{G^xG^z}Q(x,z)dt$$
$$+ [\widehat{u^xu^zh_t} - \widehat{u^xu^z}\hat{h}_t]R^{-2}[dy-\hat{h}_tdt].$$

Equation (4.25) may be rewritten by (4.2) and (2.6) as

$$\text{(A.4)} \qquad d\hat{u}(t,x) = \{\hat{\mathcal{F}}^x + [\widehat{u^xh_t} - \hat{u}^x\hat{h}_t]R^{-2}[h_t-\hat{h}_t]\}dt$$
$$+ [\widehat{u^xh_t} - \hat{u}^x\hat{h}_t]R^{-1}dv.$$

The same procedure is applicable in deriving the version of $d\hat{u}(t,z)$. Thus an application of the Itô's formula to compute $\hat{u}^x\hat{u}^z$ gives

$$\text{(A.5)} \qquad d(\hat{u}^x\hat{u}^z) = \hat{u}^z\{\hat{\mathcal{F}}^xdt + [\widehat{u^xh_t} - \hat{u}^x\hat{h}_t]R^{-2}[dy-\hat{h}_tdt]\}$$
$$+ \hat{u}^x\{\hat{\mathcal{F}}^zdt + [\widehat{u^zh_t} - \hat{u}^z\hat{h}_t]R^{-2}[dy-\hat{h}_tdt]\}$$
$$+ [\widehat{u^xh_t} - \hat{u}^x\hat{h}_t]R^{-2}[\widehat{u^zh_t} - \hat{u}^z\hat{h}_t]dt.$$

Combining (A.3) and (A.5) with (A.1), we obtain

$$\text{(A.6)} \qquad dP(t,x,z) = (\widehat{u^z\mathcal{F}^x} - \hat{u}^z\hat{\mathcal{F}}^x)dt + (\widehat{u^x\mathcal{F}^z} - \hat{u}^x\hat{\mathcal{F}}^z)dt$$
$$+ \widehat{G^xG^z}Q(x,z)dt + [\widehat{u^xu^zh_t} - \widehat{u^xu^z}\hat{h}_t$$
$$-\hat{u}^z\widehat{u^xh_t} + \hat{u}^z\hat{u}^x\hat{h}_t - \hat{u}^x\widehat{u^zh_t} + \hat{u}^x\hat{u}^z\hat{h}_t]R^{-2}[dy-\hat{h}_tdt] -$$

$$- [\widehat{u^x h_t} - \hat{u}^x \hat{h}_t] R^{-2} [\widehat{u^z h_t} - \hat{u}^z \hat{h}_t] dt.$$

In the linear case, it follows that

$$\text{(A.7)} \qquad dP(t,x,z) = (\widehat{u^z L_x u^x} - \hat{u}^z L_x \hat{u}^x) dt + (\widehat{u^x L_z u^z} - \hat{u}^x L_z \hat{u}^z) dt$$

$$+ \widehat{G^x G^z} Q(x,z) dt + [\widehat{u^x u^z \int_D H^\xi u^\xi d\xi} - \widehat{u^x u^z} \int_D H^\xi \hat{u}^\xi d\xi$$

$$- \hat{u}^z \widehat{u^x \int_D H^\xi u^\xi d\xi} + \hat{u}^z \hat{u}^x \int_D H^\xi \hat{u}^\xi d\xi - \hat{u}^x \widehat{u^z \int_D H^\xi u^\xi d\xi}$$

$$- \hat{u}^x \hat{u}^z \int_D H^\xi \hat{u}^\xi d\xi] R^{-2} [dy - \int_D H^\xi \hat{u}^\xi d\xi dt]$$

$$- [\widehat{u^x \int_D H^\xi u^\xi d\xi} - \hat{u}^x \int_D H^\xi \hat{u}^\xi d\xi] R^{-2} [\widehat{u^z \int_D H^\xi u^\xi d\xi} - \hat{u}^z \int_D H^\xi \hat{u}^\xi d\xi] dt$$

$$= (L_x \widehat{u^z u^x} - L_x \hat{u}^z \hat{u}^x) dt + (L_z \widehat{u^x u^z} - L_z \hat{u}^x \hat{u}^z) dt$$

$$+ \widehat{G^x G^z} Q(x,z) dt + [\int_D H^\xi (\widehat{u^x u^\xi} - \hat{u}^x \hat{u}^\xi) d\xi] R^{-2} [\int_D H^\xi (\widehat{u^z u^\xi} - \hat{u}^z \hat{u}^\xi) d\xi]$$

$$+ (\int_D H^\xi [\widehat{u^x u^z u^\xi} - \widehat{u^x u^z} \hat{u}^\xi - \hat{u}^z \widehat{u^x u^\xi} + \hat{u}^z \hat{u}^x \hat{u}^\xi - \hat{u}^x \widehat{u^z u^\xi} + \hat{u}^x \hat{u}^z \hat{u}^\xi] d\xi) R^{-2}$$

$$\times [dy - (\int_D H^\xi \hat{u}^\xi d\xi) dt]$$

$$= L_x P(t,x,z) dt + L_z P(t,x,z) dt + \widehat{G^x G^z} Q(x,z) dt$$

$$- [\int_D H(t,\xi) P(t,x,\xi) d\xi] R^{-2}(t) [\int_D H(t,\xi) P(t,\xi,z) d\xi] dt.$$

(Q.E.D.)

APPENDIX B. Proof of Lemma 5.3.

Define

$$\text{(B.1)} \qquad \zeta_i(t) \triangleq \int_0^t \hat{h}_i(s,u_s) R^{-2}(s) dy(s) - \frac{1}{2} \int_0^t \hat{h}_i^2(s,u_s) R^{-2}(s) ds.$$

Then, (5.18) is expressed as

$$\text{(B.2)} \qquad \Lambda_i(t) = \exp\{\zeta_i(t)\}.$$

Noting from (B.1) that the $\zeta_i(t)$-process has the stochastic differential,

$$\text{(B.3)} \qquad d\zeta_i(t) = \hat{h}_i(t,u_t)R^{-2}(t)dy(t) - \frac{1}{2}\hat{h}_i^{\,2}(t,u_t)R^{-2}(t)dt,$$

and applying Itô's chain rule to the function Λ_i, it follows that

$$\text{(B.4)} \qquad d\Lambda_i = \frac{\partial\Lambda_i}{\partial t}dt + \frac{\partial\Lambda_i}{\partial\zeta_i}d\zeta_i + \frac{1}{2}\frac{\partial^2\Lambda_i}{\partial\zeta_i^{\,2}}(d\zeta_i)^2.$$

It is a simple exercise to show that

$$\text{(B.5)} \qquad \frac{\partial\Lambda_i}{\partial t} = 0, \quad \frac{\partial\Lambda_i}{\partial\zeta_i} = \Lambda_i, \quad \frac{\partial^2\Lambda_i}{\partial\zeta_i^{\,2}} = \Lambda_i$$

and that

$$\text{(B.6)} \qquad (d\zeta_i)^2 = \hat{h}_i^{\,2}(t,u_t)R^{-2}(t)dt,$$

where (5.8) has been used to derive (B.6). By substituting (B.5) and (B.6) into (B.4), the proof has been completed. (Q.E.D.)

APPENDIX C. Proof of Theorem 5.2.

By applying the Itô's chain rule to (5.24), we have

$$\text{(C.1)} \qquad dM_i(t) = \frac{\partial M_i}{\partial t}dt + \sum_{j=1}^{K}\frac{\partial M_i}{\partial\Lambda_{ji}}\frac{\partial\Lambda_{ji}}{\partial\zeta_{ji}}d\zeta_{ji}$$

$$+ \frac{1}{2}\sum_{j=1}^{K}\sum_{k=1}^{K}\frac{\partial M_i}{\partial\zeta_{ji}\partial\zeta_{ki}}d\zeta_{ji}d\zeta_{ki},$$

where the $\zeta_{ji}(t)$-process is defined by

$$\text{(C.2)} \qquad \zeta_{ji}(t) \triangleq \left[\int_0^t\hat{h}_j(s,u_s)R^{-2}(s)dy(s) - \frac{1}{2}\int_0^t\hat{h}_j^{\,2}(s,u_s)R^{-2}(s)ds\right]$$

$$- \left[\int_0^t\hat{h}_i(s,u_s)R^{-2}(s)dy(s) - \frac{1}{2}\int_0^t\hat{h}_i^{\,2}(s,u_s)R^{-2}(s)ds\right]$$

and this has the stochastic differential,

$$\text{(C.3)} \qquad d\zeta_{ji}(t) = (\hat{h}_j-\hat{h}_i)R^{-2}dy - \frac{1}{2}(\hat{h}_j^{\,2}-\hat{h}_i^{\,2})R^{-2}dt.$$

In (C.1), it can be shown from (5.24) that

$$\text{(C.4)} \qquad \frac{\partial M_i}{\partial t} = 0, \quad \frac{\partial M_i}{\partial \Lambda_{ji}} = -\alpha_{ji} M_i^{\,2}, \quad \frac{\partial \Lambda_{ji}}{\partial \zeta_{ji}} = \Lambda_{ji}$$

and

$$\text{(C.5a)} \qquad \frac{\partial^2 M_i}{\partial \zeta_{ji} \partial \zeta_{ki}} = 2\alpha_{ji}\alpha_{ki}\Lambda_{ji}\Lambda_{ki}M_i^{\,3} \qquad \text{for } j \neq k$$

$$\text{(C.5b)} \qquad = -\alpha_{ji}\Lambda_{ji}M_i^{\,2} + 2\alpha_{ji}^{\,2}\Lambda_{ji}^{\,2}M_i^{\,3} \qquad \text{for } j=k.$$

Substituting (C.4) and (C.5) into (C.1), we have

$$\text{(C.6)} \qquad dM_i = -\sum_{j=1}^{K} \alpha_{ji}\Lambda_{ji}M_i^{\,2} d\zeta_{ji} - \frac{1}{2}\sum_{j=1}^{K} \alpha_{ji}\Lambda_{ji}M_i^{\,2}(d\zeta_{ji})^2$$

$$+ \sum_{j=1}^{K}\sum_{k=1}^{K} \alpha_{ji}\alpha_{ki}\Lambda_{ji}\Lambda_{ki}M_i^{\,3}(d\zeta_{ji})(d\zeta_{ki}).$$

From (C.3) it follows that

$$\text{(C.7)} \qquad (d\zeta_{ji})^2 = (\hat{h}_j - \hat{h}_i)^2 R^{-2} dt$$

$$\text{(C.8)} \qquad (d\zeta_{ji})(d\zeta_{ki}) = (\hat{h}_j - \hat{h}_i)(\hat{h}_k - \hat{h}_i) R^{-2} dt.$$

Hence the substitution of (C.3), (C.7) and (C.8) into (C.6) yiels (5.35).
(Q.E.D.)

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